MATHEMATICS MAGAZINE

CONTENTS

CODEN: MAMGA8

VOLUME 48 • MARCH 1975 • NUMBER 2

THE MATHEMATICAL ASSOCIATION OF AMERICA
1225 Connecticut Avenue, N.W.
Washington, DC 20036

Address Change Requested

MM

MATHEMATICS MAGAZINE

GERHARD N. WOLLAN, *EDITOR*

EDITORIAL CORRESPONDENCE until May 1, 1975, should be sent to the EDITOR, G. N. WOLLAN, Department of Mathematics, Purdue University, Lafayette, Indiana 47907. After May 1, 1975, editorial correspondence should be sent to the editors-elect, J. ARTHUR SEEBACH JR. and LYNN A. STEEN, Department of Mathematics, St. Olaf College, Northfield, Minnesota 55057. Authors should submit two copies and keep one copy for protection against possible loss of the manuscript. Figures should be drawn on separate sheets in India ink. They should be appropriately lettered and of a suitable size for photographing.

REPRINT PERMISSION should be requested from LEONARD GILLMAN, Mathematical Association of America, University of Texas, Austin, Texas 78712. (See also the copyright notice below.)

NOTICE OF CHANGE OF ADDRESS and other subscription correspondence should be sent to the Executive Director, A. B. WILLCOX, Mathematical Association of America, Suite 310, 1225 Connecticut Avenue, N. W., Washington, D. C. 20036.

ADVERTISING CORRESPONDENCE should be addressed to RAOUL HAILPERN, Mathematical Association of America, SUNY at Buffalo (University of Buffalo), Buffalo, New York 14214.

THE MATHEMATICS MAGAZINE is published by the Mathematical Association of America at Washington, D.C., bi-monthly except July-August. Ordinary subscriptions are $10 per year. Members of the Mathematical Association of America and of Mu Alpha Theta may subscribe at the special rate of $7.00. Back issues may be purchased, when in print, for $2.00.

Second class postage paid at Washington, D.C. and additional mailing offices.

COVERING DELETED CHESSBOARDS WITH DOMINOES

DAVID SINGMASTER, Polytechnic of the South Bank, London, and
Istituto Matematico, Pisa, Italy

1. Introduction. In the January, 1973, issue, Richard Gibbs [1] has restated the well-known fact that a $2m \times 2m$ chessboard, with two diagonally opposite corners deleted, cannot be covered by dominoes since the deleted squares are of the same color. He then asked:

(a) Can a $2m \times 2m$ board be covered with dominoes if we remove any two squares of opposite color?

(b) Can a $(2m + 1) \times (2m + 1)$ board be covered if we remove any square of the major (or corner) color?

It is well known and easily seen that an $m \times n$ rectangle can be covered with dominoes if and only if m or n is even. This led me to consider Gibbs' problem (a) for $m \times 2n$ boards and thence to consider his problem (b) for $(2m + 1) \times (2n + 1)$ boards. Theorems 1 and 3 below answer these generalized problems affirmatively, except for the case when $m = 1$ in (a), in which case a further necessary and sufficient condition for covering is given as Theorem 2.

It is not difficult to see that (a) fails if we delete four squares, but I was rather delighted to find that (b) is also true if we remove any three squares, two of the major color and one of the minor, provided $m \neq 0$ and $n \neq 0$. This is proven as Theorem 4. The extension of (b) to five squares fails.

These results have further valid extensions to n dimensions. Theorem 5 asserts that an n-dimensional board of dimensions $m_1 \times m_2 \times \cdots \times m_n$, where some m_i is even and none are one, with $2n - 2$ cells deleted, $n - 1$ of each color, can be covered with n-dimensional dominoes. Theorem 6 gives the analogous result when all m_i are odd and with $2n - 1$ cells deleted, n of the major color and $n - 1$ of the minor color. These results are the best possible, in that deletion of more cells can leave an uncoverable board. Theorems 5 and 6 supersede Theorems 1, 3 and 4, but I have retained the separate proofs of Theorems 1, 3 and 4 as they are much different from the general proofs.

Since writing this note, two solutions of Gibbs' problems have appeared as [2]. The first solution applies only to square boards. The second solution, by C. W. Trigg, clearly generalizes to the rectangular cases treated in my Theorems 1 and 3. However, my solutions are quite different from both of the published solutions and I introduce concepts in the proofs which I use for later results, so I have retained my solutions.

2. The two dimensional case.

THEOREM 1. *For $m > 1$, an $m \times 2n$ chessboard, with any two squares of opposite color deleted, can be covered with dominoes.*

Proof. We shall give diagrams for four cases. In each diagram the deleted squares are both marked with a cross and the remainder of the board is divided into rectangles of which each have an even side. The even direction is indicated by $\leftrightarrow$ and its length is given.

We orient the board with its even direction along the x-axis and we label the squares (x, y) with $1 \leqq x \leqq 2n$ and $1 \leqq y \leqq m$. We denote the colors as 0 and 1 and we say the color of the square (x, y) is congruent to $x + y \pmod 2$. Henceforth, all congruences are taken (mod 2). Let the deleted squares be (a, b) and (c, d). We choose $a \leqq c$, and by reflecting if necessary, we can also choose $b \leqq d$. Since (a, b) and (c, d) have different colors, we have $a + b \not\equiv c + d$. Hence either $a \equiv c$, $b \not\equiv d$ or $a \not\equiv c$, $b \equiv d$.

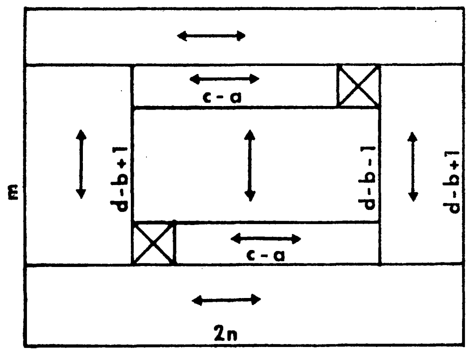


FIG. 1

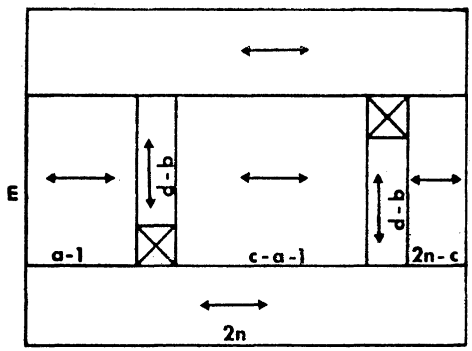


FIG. 2

Case 1. $a \equiv c$, $b \not\equiv d$. Then $d - b \pm 1$ and $c - a$ are even and Figure 1 gives a covering.

Case 2-A. $a \not\equiv c$, $b \equiv d$, a odd. Then $a - 1$, $c - a - 1$, $2n - c$ and $d - b$ are even, so Figure 2 gives a covering.

Case 2-B-1. $a \not\equiv c$, $b \equiv d$, a even, $b \neq d$. Then $a \neq 1$, $c \neq 2n$ and $a - 2$, $c - a + 1$, $2n - c - 1$ and $d - b$ are even, thus Figure 3 gives a covering.

Case 2-B-2. $a \not\equiv c$, $b = d$, a even. Then $a \neq 1$, $c \neq 2n$ and $a - 2$, $c - a \pm 1$ and $2n - c - 1$ are even, so Figure 4 gives a covering when $b < m$. If $b = m$, then we reflect Figure 4 for a covering. ∎

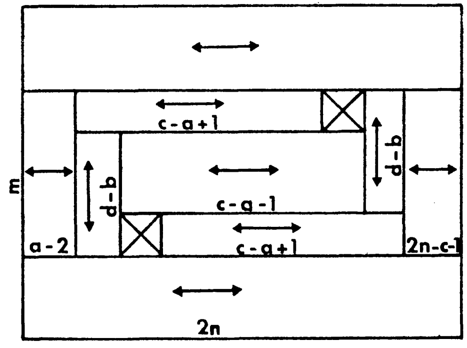


FIG. 3

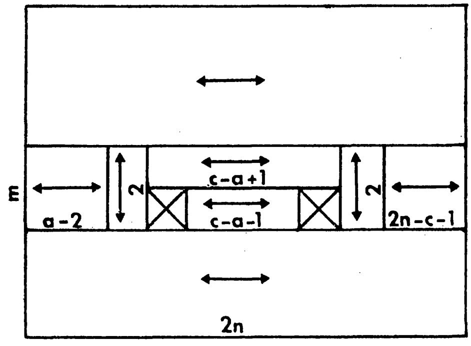


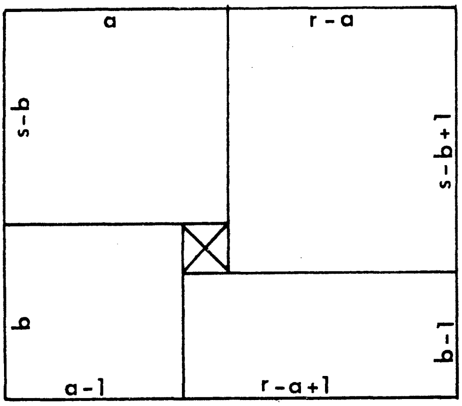
FIG. 4

If $m = 1$, then we have $b = d = 1$. Case 2–A is still valid, but Case 2–B–2 fails and one easily sees that no covering is then possible. We can thus assert the following:

THEOREM 2. *Consider a strip of $2n$ squares, numbered $1, 2, \cdots, 2n$ and alternately colored. Suppose the ath and cth squares are deleted, where $a < c$ and $a \not\equiv c$ (i.e., the squares are differently colored). Then the deleted strip can be covered with dominoes if and only if a is odd.*

THEOREM 3. *For any odd integers r and s, an $r \times s$ chessboard, with any one square of the major color (i.e., the corner color) deleted, can be covered with dominoes.*

Proof. Consider Figure 5. Letting the deleted square be (a, b), we have $a + b \equiv 0$. If a and b are both even, then so are $r - a + 1$ and $s - a + 1$. If a and b are both odd, then $a - 1$, $b - 1$, $r - a$ and $s - b$ are all even. In either case, all four rectangles in the figure have an even side. ▮



FIG, 5

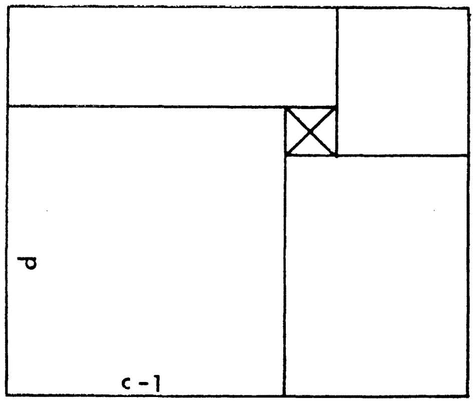


FIG. 6

THEOREM 4. *For any odd integers r and s, both greater than one, an $r \times s$ chessboard, with any three squares deleted, two of the major color and one of the minor color, can be covered with dominoes.*

Proof. Let the deleted squares be (a, b), (c, d) and (e, f) with $a + b \equiv c + d \equiv 0$ and $e + f \not\equiv 0$. As in Theorem 1, we may assume $a \leqq c$, $b \leqq d$.

First we show that we may assume $c = a + 1$, $d = b + 1$. Suppose $c \geqq a + 2$. Suppose further that $e \neq a + 1$. Then the division of the $r \times s$ board into the first a (or $a + 1$) columns and the remaining $r - a$ (or $r - a - 1$) gives two rectangles to which Theorems 1 (or 2) and 3 can be applied to produce a covering. Suppose now $e = a + 1$. If $f \geqq b$, then (c, d) and (e, f) both lie in the upper right rectangle of Figure 5. We can apply Theorem 1 (or 2 if $f = b$) to produce a covering of this rectangle and the other rectangles all have an even side, so we have a covering of the board. If $f < b$, consider Figure 6, which is Figure 5 drawn for (c, d). Both (a, b) and

(e, f) are in the lower left rectangle, so the argument just used gives a covering. (Indeed this second case is just a 180° rotation of the first.) Hence we need only consider $a \leqq c \leqq a+1$. Symmetrically, we need only consider $b \leqq d \leqq b+1$ and now $a+b \equiv c+d$ gives $c = a+1$, $d = b+1$.

Next we show that we may assume $e \leqq a, f > b$ or $e > a, f \leqq b$. Suppose $e \leqq a$. If $f \leqq b$, then both (a, b) and (e, f) are in the lower left rectangle of Figures 6 and 8. Theorem 1 can be applied to at least one of these diagrams to produce a covering as above. Now suppose $e > a$. If $f > b$, then both (c, d) and (e, f) are in the upper right rectangle of Figures 5 and 7 and we have a covering by the argument just used.

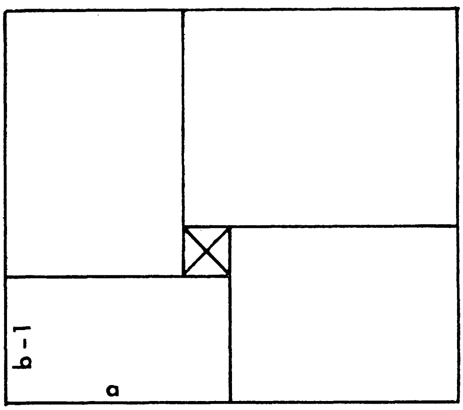


FIG. 7

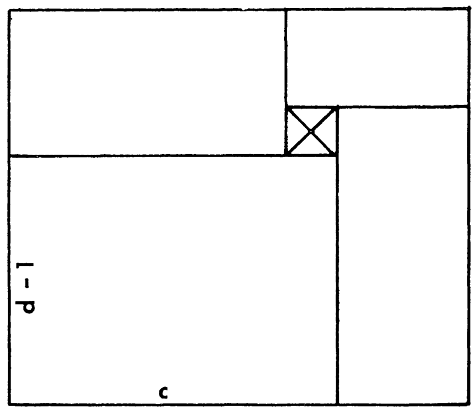


FIG. 8

Suppose now that $e \leqq a, f > b$. We know $a+b \equiv 0$. If a and b are even, then the division of the board into the first a columns and the remaining $r-a$ gives two rectangles to which Theorems 1 and 3 can be applied and we have a covering. If a and b are odd, then the division into the first b rows and the remaining $s-b$ gives two rectangles to which Theorems 3 and 1 apply. The case $e > a, f \leqq b$ can be similarly treated and we have now shown that there is a covering in any situation. ▮

If r or s is one, then Theorem 4 fails. One can write down the necessary and sufficient conditions for covering in this case but they are not very interesting. The extension of Theorem 1 to deletion of four squares and the extension of Theorem 4 to deletion of five squares both fail since one can isolate a single corner square in either case.

3. The n-dimensional case. First let us introduce some terms. We say that an n-dimensional board, with dimensions $m_1 \times m_2 \times \cdots \times m_n$, is: *proper* if all m_i are greater than one; *even* if some m_i is even (i.e., it has even volume); *odd* if all m_i are odd. An *n-domino* or *n-dimensional domino* is a $1 \times 1 \times \cdots \times 1 \times 2$ block. Generally we shall just call this a domino. It is easily seen that a board can be covered with dominoes if and only if the board is even. We shall call the unit n-cubes of the board, *cells*.

THEOREM 5. *A proper even n-dimensional board, with any $2n-2$ cells deleted, half of each color, can be covered with dominoes.*

Proof. We label the cells as $(x_1, x_2, \cdots, x_n)$ with $1 \leqq x_i \leqq m_i$. The standard 0-1 coloring is given by color $(x_1, x_2, \cdots, x_n) \equiv \Sigma x_i$. We shall refer to the colored cells as 0–cells and 1–cells. Note that we can reverse the coloring without changing the basic situation. Assume that $2 \mid m_n$.

The proof proceeds by induction on n. For each n, we describe two covering processes which allow us to reduce the problem to the case when all m_i are 2. In this case, we can use the processes to cover the cube unless all deleted 1–cells have $x_1 = x_2 = \cdots = x_n = 1$ and this is either impossible or a trivial case. Now to the details.

For $n = 1$, the theorem is trivially true. Assume $n > 1$ and that the theorem holds for $n - 1$. Consider the $(n-1)$-dimensional layer or (hyper-)plane of cells determined by $x_1 = 1$. Let the number of deleted 0–cells (1–cells) in this layer be $k_0(k_1)$. By reversing colors, if necessary, we may assume $k_0 \leqq k_1$.

Case 1. $k_1 \leqq n-2$. Let $d = k_1 - k_0$. We want to find d undeleted 0–cells with $x_1 = 1$ such that the adjacent 1–cell with $x_1 = 2$ is also undeleted. Let $A = m_2 \cdot m_3 \cdot \cdots m_n$. We have $A \geqq 2^{n-1} \geqq 2(n-1)$ for $n \geqq 1$. There are $A/2 - k_0$ undeleted 0–cells with $x_1 = 1$ and there are at most $n - 1 - k_1$ deleted 1–cells with $x_1 = 2$. Thus there are at most $A/2 - k_0 - n + 1 + k_1 = A/2 - n + 1 + d \geqq d$ undeleted 0–cells with $x_1 = 1$ having undeleted adjacent 1–cells with $x_1 = 2$.

Place d dominoes in d pairs of these cells. The layer with $x_1 = 1$ can now be considered as having k_1 1–cells and k_1 0–cells deleted, where $k_1 \leqq n-2$. By the inductive hypothesis, we can cover this layer with dominoes. (Here we are using the natural equivalence of the $(n-1)$-dimensional problem for $m_2 \times m_3 \times \cdots \times m_n$ with the n-dimensional problem for $1 \times m_2 \times m_3 \times \cdots \times m_n$.) The remaining part of the board can be considered as having $n - 1 - k_0$ 0–cells and $n - 1 - k_0$ 1–cells deleted, so we have reduced the problem to an $(m_1 - 1) \times m_2 \cdots \times m_n$ board. Here and elsewhere, we are using the fact that the statement of the theorem includes, *a fortiori*, the deletion of fewer than $2n - 2$ cells, provided we delete as many of each color.

Case 2-A. $k_1 = n-1$, $k_0 < n-1$. Choose any k_0 1–cells in the plane $x_1 = 1$ and consider this plane as having these k_0 1–cells and the k_0 0–cells deleted. Since $k_0 < n-1$, the inductive hypothesis gives a covering of this plane. Let $d = k_1 - k_0 = n - 1 - k_0$. There are d 1–cells which have been covered by this covering, but which we wish to have deleted. Consider the domino covering one of these. It covers an adjacent 0–cell. Turn the domino so its length is in the x_1 direction and covers this adjacent 0–cell and its 1–cell neighbor having $x_1 = 2$. This neighbor cannot be a deleted 1–cell since all deleted 1–cells have $x_1 = 1$. We carry this out for all the d 1–cells which were covered and which we want to be deleted. The remaining portion of the board, with $x_1 \geqq 2$, can be considered as having $n - 1 - k_0$ 0–cells and $n - 1 - k_0$ 1–cells deleted, so we have again reduced the problem to an $(m_1 - 1) \times m_2 \times \cdots \times m_n$ board.

Case 2-B. $k_1 = n - 1, k_0 = n - 1$. Then all the deleted cells are in the first layer and we may easily cover the layers with $x_1 = m_1, x_1 = m_1 - 1, \cdots, x_1 = 3$, thus reducing the problem to a $2 \times m_2 \times \cdots \times m_n$ board.

By repeatedly applying these reductions, we eventually reduce the problem to the $2 \times m_2 \times \cdots \times m_n$ board. We can then apply the reduction to each other dimension, thus reducing the problem to the $2 \times 2 \times \cdots \times 2$ board (i.e., the n-cube).

On this n-cube, again consider the layer with $x_1 = 1$, let $k_0(k_1)$ be the number of deleted 0–cells (1–cells) in this layer and assume $k_0 \leqq k_1$. By reflecting the cube, if necessary, we can assume $k_1 \neq 0$.

Case 3-A. $k_1 \leqq n - 2$, $k_0 = 0$. The plane with $x_1 = 2$ has $j_0 = n - 1$ deleted 0–cells and $j_1 = n - 1 - k_1$ deleted 1–cells. We can apply the argument of Case 2–A to this by interchanging 0–cells with 1–cells and $x_1 = 1$ with $x_1 = 2$ throughout the argument. This gives a covering of the $x_1 = 2$ plane and leaves the $x_1 = 1$ plane with k_1 0–cells and k_1 1–cells deleted. By the inductive hypothesis, we can cover this layer and we are done.

Case 3-B. $k_1 \leqq n - 2$, $k_0 > 0$. The argument of Case 1 applies directly to give a covering of the $x_1 = 1$ layer. The $x_1 = 2$ layer is left with $n - 1 - k_0$ 0–cells and $n - 1 - k_0$ 1–cells deleted. By the inductive hypothesis, this can also be covered and we are done.

Thus we can cover our deleted n-cube unless $k_1 = n - 1$, i.e., all deleted 1-cells have $x_1 = 1$. We can apply this argument in each of the remaining dimensions, so we have a covering unless all deleted 1–cells have $x_1 = 1 = x_2 = \cdots = x_n$. But there is only one such cell in the n-cube and we have $n - 1$ deleted 1–cells. For $n > 2$, this is impossible. For $n = 2$, the 2×2 problem is simple. ▮

THEOREM 6. *A proper odd n-dimensional board, with any* $2n - 1$ *cells deleted, n of the major color and* $n - 1$ *of the minor color, can be covered with dominoes.*

Proof. Assume the colors are assigned so that the corners are colored 0, so that we are deleting n 0–cells and $n - 1$ 1–cells. The pattern of proof is similar to that of Theorem 5. We describe one covering process which allows us to reduce to the $3 \times 3 \times \cdots \times 3$ case and then to solve that case.

For $n = 1$ (and $n = 2$) the theorem is true. Assume $n > 1$ and that the theorem is valid for $n - 1$. Consider the two layers given by $x_1 = 1$ or $x_1 = 2$. Let the number of deleted 0–cells (1–cells) in these layers be $k_0(k_1)$.

Case 1. $k_0 \leqq n - 1$. At this point, the arguments for $k_0 \leqq k_1$ and for $k_1 \leqq k_0$ are symmetric, so we consider just $k_0 \leqq k_1$. Let $d = k_0 - k_1$. By the argument used in Case 1 of Theorem 5, we can find d undeleted 0–cells with $x_1 = 2$ such that the adjacent 1–cell with $x_1 = 3$ is also undeleted. Placing d dominoes in these d pairs of cells, the first two layers can be considered as a $2 \times m_2 \times \cdots \times m_n$ board with k_1 1–cells and k_1 0–cells deleted. Since $k_1 \leqq n - 1$, we can cover these layers by Theorem 5. The remaining part of the board, with $x_1 \geqq 3$, can now be considered as having $n - k_0$ 0–cells and $n - 1 - k_0$ 1–cells deleted, so we have reduced the problem to an $(m_1 - 2) \times m_2 \times \cdots \times m_n$ board.

Case 2. $k_0 = n$. Now all deleted 0–cells are in the first two layers. So the last two layers (with $x_1 = m_1$ or $x_1 = m_1 - 1$) contain no deleted 0–cells, if $m_1 > 3$. The argument just used for Case 1 can be applied to cover these layers and reduce the problem to an $(m_1 - 2) \times m_2 \times \cdots \times m_n$ board.

By repeated application of this reduction, we can reduce the problem to the $3 \times m_2 \times \cdots \times m_n$ board and thence to the $3 \times 3 \times \cdots \times 3$ board. In this situation, we again consider the first two layers, $x_1 = 1$ or $x_1 = 2$, and the numbers k_0 and k_1, as before. By reflection, we may assume $k_0 > 0$.

Case 3-A. $k_0 \leqq n - 1$, $k_1 = 0$. Then the last two layers, with $x_1 = 2$ or $x_1 = 3$, have at least $j_0 = n - k_0$ deleted 0–cells and $j_1 = n - 1$ deleted 1–cells. Since $k_0 > 0$, we have $j_0 \leqq n - 1$ and the argument of Case 1 gives a covering of the last two layers, leaving the first layer with k_0 deleted 0–cells and $k_0 - 1$ deleted 1–cells. By the inductive hypothesis, we can cover the first layer.

Case 3-B. $k_0 \leqq n - 1$, $k_1 > 0$. The process given in Case 1 gives a covering of the first two layers, leaving the third layer with $n - \min(k_0, k_1)$ deleted 0–cells and $n - 1 - \min(k_0, k_1)$ deleted 1–cells. Since $\min(k_0, k_1) > 0$, we can apply the inductive hypothesis to cover the third layer.

Hence we can cover our deleted $3 \times 3 \times \cdots \times 3$ cube unless $k_0 = n$, i.e., all deleted 0–cells are in the first two layers. Now consider the last two layers, with $x_1 = 2$ or $x_1 = 3$, and let j_0 be the number of deleted 0–cells in these layers. If $j_0 > 0$, we can proceed as in Case 3 and we obtain a covering of the deleted board unless $j_0 = n$. But if $j_0 = n$ and $k_0 = n$, then all the deleted 0–cells must have $x_1 = 2$. On the other hand, if $j_0 = 0$ and $k_0 = n$, then all the deleted 0–cells must have $x_1 = 1$. Let $y_1 = 2$ or 1 according to which of these situations occurs. Then we have that all the deleted 0–cells have $x_1 = y_1$. (By more detailed argument, one can suppose $y_1 = 1$.)

We now apply this process to each other dimension, obtaining a covering unless all deleted 0–cells have $x_1 = y_1, x_2 = y_2, \cdots, x_n = y_n$. But there is at most one such cell and we are deleting n 0–cells, with $n > 1$, so this last case is impossible. ∎

It is easy to see that Theorems 5 and 6 cannot be extended to deletion of more cells since the deletion of n cells of the same color (or of the minor color) can isolate a corner cell.

4. Extensions and conjectures. The extension of these results to coverings by straight trominoes or straight k-ominoes, with $k \geqq 3$, seems beset with complications, even if we consider only the standard k-coloring with the color of $(x_1, x_2, \cdots, x_n)$ congruent to $\Sigma x_i \pmod k$. For example, the $2 \times 2 \times \cdots \times 2 \times k$ board can be covered with straight k-ominoes in only one way and almost any deletion of k cells, one of each color, leaves an uncoverable board. Presumably, one must hypothesize that all $m_i \geqq k$.

Further, in the standard k-coloring of an n-dimensional board, with $k \geqq 3$, the n neighbors of a corner cell have just two colors and some corners have $(n + 1)/2$ of one color and $n/2$ of another, hence one can delete at most $(n - 1)/2$ of each color and still hope for a covering. Perhaps the following analog of Theorem 5 is true:

CONJECTURE A. *Let $k \geqq 3$ and consider an $m_1 \times m_2 \times \cdots \times m_n$ board with $k \mid m_n$ and all $m_i \geqq k$. If we delete any $k(n-1)/2$ cells, $(n-1)/2$ of each color, then the remainder of the board can be covered with straight k-ominoes.*

The formulation of the analogs of Theorems 3, 4 and 6 is somewhat more complicated because of the varying number of cells to be deleted for each color. For an $m_1 \times m_2 \times \cdots \times m_n$ board with the standard coloring, let a_i be the number of cells of color i. Let $b = \min\{a_i\}$ and let $d_i = a_i - b$. Then perhaps the following analog of Theorem 6 is correct:

CONJECTURE B. *Let $k \geqq 3$ and consider an $m_1 \times m_2 \times \cdots \times m_n$ board with all $m_i \geqq k$. Let d_i be as just defined. If, for each color i, we delete any $(n-1)/2 + d_i$ cells of color i, then the remainder of the board can be covered with straight k-ominoes.*

Note that Conjecture B includes Conjecture A as a special case. The analog of Theorem 3 is Conjecture B with only d_i cells of color i deleted.

Acknowledgement. This work was supported by an Italian National Research Council (CNR) research fellowship.

References

1. R. A. Gibbs, Problem 852, this MAGAZINE, 46 (1973) 42.
2. Michael Gilpin and C. W. Trigg, Solutions of Problem 852, this MAGAZINE, 46 (1973) 287–8.

CONTINUITY OF INVERSE FUNCTIONS

MICHAEL J. HOFFMAN, University of California, Berkeley

I. When does a continuous bijection have a continuous inverse? One positive result is that a continuous injection of a compact space into a Hausdorff space is a homeomorphism onto its image. A popular counterexample is a map of a half-open interval onto a circle by an exponential function. Simpler ones are available, including real valued functions defined on subsets of the real line. For what sets is this kind of behavior impossible? We present necessary and sufficient conditions on a subset B of the real line $\mathbf{R}$ such that every continuous injection of B into $\mathbf{R}$ has a continuous inverse on $f(B)$. All sets are to be contained in $\mathbf{R}$ and carry the relative topology. Intervals are indicated by listing the endpoints between brackets, square if the end is to be included, round if it is not.

DEFINITION. *A subset B of $\mathbf{R}$ is **stable** if every continuous injection of B into $\mathbf{R}$ is a homeomorphism onto its range.*

DEFINITION. *If $B \subseteq \mathbf{R}$, the intersection of B with its boundary will be denoted by B^*. A point $x \in B^*$ is screened if the component of B containing x is an interval and B contains points arbitrarily close to x on both sides.*

CONJECTURE A. *Let $k \geqq 3$ and consider an $m_1 \times m_2 \times \cdots \times m_n$ board with $k \mid m_n$ and all $m_i \geqq k$. If we delete any $k(n-1)/2$ cells, $(n-1)/2$ of each color, then the remainder of the board can be covered with straight k-ominoes.*

The formulation of the analogs of Theorems 3, 4 and 6 is somewhat more complicated because of the varying number of cells to be deleted for each color. For an $m_1 \times m_2 \times \cdots \times m_n$ board with the standard coloring, let a_i be the number of cells of color i. Let $b = \min\{a_i\}$ and let $d_i = a_i - b$. Then perhaps the following analog of Theorem 6 is correct:

CONJECTURE B. *Let $k \geqq 3$ and consider an $m_1 \times m_2 \times \cdots \times m_n$ board with all $m_i \geqq k$. Let d_i be as just defined. If, for each color i, we delete any $(n-1)/2 + d_i$ cells of color i, then the remainder of the board can be covered with straight k-ominoes.*

Note that Conjecture B includes Conjecture A as a special case. The analog of Theorem 3 is Conjecture B with only d_i cells of color i deleted.

Acknowledgement. This work was supported by an Italian National Research Council (CNR) research fellowship.

References

1. R. A. Gibbs, Problem 852, this MAGAZINE, 46 (1973) 42.
2. Michael Gilpin and C. W. Trigg, Solutions of Problem 852, this MAGAZINE, 46 (1973) 287–8.

CONTINUITY OF INVERSE FUNCTIONS

MICHAEL J. HOFFMAN, University of California, Berkeley

I. When does a continuous bijection have a continuous inverse? One positive result is that a continuous injection of a compact space into a Hausdorff space is a homeomorphism onto its image. A popular counterexample is a map of a half-open interval onto a circle by an exponential function. Simpler ones are available, including real valued functions defined on subsets of the real line. For what sets is this kind of behavior impossible? We present necessary and sufficient conditions on a subset B of the real line $\boldsymbol{R}$ such that every continuous injection of B into $\boldsymbol{R}$ has a continuous inverse on $f(B)$. All sets are to be contained in $\boldsymbol{R}$ and carry the relative topology. Intervals are indicated by listing the endpoints between brackets, square if the end is to be included, round if it is not.

DEFINITION. *A subset B of $\boldsymbol{R}$ is **stable** if every continuous injection of B into $\boldsymbol{R}$ is a homeomorphism onto its range.*

DEFINITION. *If $B \subseteq \boldsymbol{R}$, the intersection of B with its boundary will be denoted by B^*. A point $x \in B^*$ is screened if the component of B containing x is an interval and B contains points arbitrarily close to x on both sides.*

DEFINITION. *B is said to be* ****-finite*** *if it satisfies either of the following two equivalent conditions*:

(1) *$B\setminus U$ has finitely many components where U is any open set of the form $U = \bigcup \{B_b : b \in B^*\}$ and $B_b = \{x : |x - b| < \delta_b\}$ for some $\delta_b > 0$.*

(2) *Any sequence of points belonging to distinct components of B of nonzero length must cluster somewhere in B.*

That is, B itself may have infinitely many components, but if some interval is deleted around each boundary point, the remainder has finitely many components. The component containing a point x will be denoted by co(x), and $B\setminus U$ denotes the set of points of B which are not in U. Notice that since the deleted sets are intersections of B with intervals centered at boundary points of B, deletion of U can only reduce the number of components, and distinct components of $B\setminus U$ are contained in distinct components of B.

Suppose (1) holds and $\{x_n\}_{n=1}^{\infty}$ are such that co(x_n) are distinct intervals. If the x_n did not cluster anywhere in B, then $\{x_n\}\setminus\{b\}$ would be bounded away from b for each $b \in B^*$. Say $|x_n - b| > \delta_b$ for each n except possibly if $x_n = b$. Let $B_b = \{x: |x - b| < \delta_b$ and $|x - b| < \frac{1}{4}$ length co(b) if the last is nonzero$\}$, and $U = \bigcup\{B_b : b \in B^*\}$. If $x_n \notin B^*$ then $x_n \in B\setminus U$; and if $x_n \in B^*$ then the midpoint of co$(x_n) \in B\setminus U$. Thus $B\setminus U$ has infinitely many components, contradicting (1).

If (1) fails, there is a set U of the form given above such that $B\setminus U$ has infinitely many components. Choose x_n from distinct components of $B\setminus U$. By a remark above, the x_n lie in distinct components of B. If they clustered at a point $b \in B$, this would force $b \in B^*$ and so $b \in U$. But the x_n cannot cluster in the open set U since none of them are in U. This shows the equivalence of (1) and (2).

Our main result is the following theorem:

THEOREM. *A subset B of **R** is stable if and only if it is in one of the following classes*:

(1) *compact sets,*

(2) *open sets,*

(3) *half-open intervals,*

(4) **-finite sets B such that every $x \in B^*$ is screened.*

II. Nonstable sets. Before proving the main theorem, we construct several counterexamples. These will show the types of things which can go wrong and guide the development of the general proofs.

(1) Let $B = [0,1) \cup (1,2]$, and put $f(x) = x$ for $x \in [0,1)$ and $f(x) = 3 - x$ for $x \in (1,2]$. The inverse is discontinuous at 1.

(2) Let $B = [-\pi/2, -1] \cup [0, \infty)$, and put $f(x) = x + \pi$ for $x \in [-\pi/2, -1]$ and $f(x) = \arctan(x)$ for $x \geqq 0$. The inverse is discontinuous at $\pi/2$. This illustrates the use we will make of arctangent later when the only missing limit points available lie at infinity.

(3) Let B be the integers and put $f(0) = 0$ and $f(n) = 1/n$ for $n \neq 0$. The inverse is discontinuous at 0.

(4) Let B_1 be the rational points in $[0,1]$ and B_2 be the irrational points in $[1,2]$. Put $f(x) = x$ for $x \in B_1$ and $f(x) = 2 - x$ for $x \in B_2$. The inverse is discontinuous at every point of $[0,1]$ except at 1.

(5) Let B be the real line with the points $1/n$ deleted for nonzero integers n. Then 0 is an unscreened point of B^*. Put $f(0) = 1$ and $f(x) = x$ for $x > 1$ and $f(x) = x + 1$ for $x < -1$. Map the remaining segments of B linearly onto segments between 0 and 1 according to the alternating pattern

$$\left(-\frac{1}{n}, -\frac{1}{n+1}\right) \quad \text{onto} \quad \left(1 - \frac{1}{2n-1}, 1 - \frac{1}{2n}\right),$$

$$\left(\frac{1}{n+1}, \frac{1}{n}\right) \quad \text{onto} \quad \left(1 - \frac{1}{2n}, 1 - \frac{1}{2n+1}\right).$$

See Figure 1a. The discontinuity of the inverse at 1 shows the kind of thing that can happen at the image of an unscreened boundary point.

(6) Our last example shows what may happen if B is not *-finite. It is very similar to the last example:

$$B = \boldsymbol{R} \setminus \{\cdots, 1/4, 1/3, 1/2, 1, 2, 3, \cdots\}.$$

Put $f(x) = 1 - x$ for $x \leqq 0$. Map the remaining segments of B linearly onto segments between 0 and 1 according to the alternating pattern

$$\left(\frac{1}{n+1}, \frac{1}{n}\right) \quad \text{onto} \quad \left(1 - \frac{1}{2n-1}, 1 - \frac{1}{2n}\right),$$

$$(n, n+1) \quad \text{onto} \quad \left(1 - \frac{1}{2n}, 1 - \frac{1}{2n+1}\right).$$

See Figure 1b. The inverse is discontinuous at 1.

III. The main theorem. The standard result that compact sets are stable may be found in [**2**, p. 141]. A continuous injection of an interval into $\boldsymbol{R}$ is strictly monotone and a homeomorphism onto its image [**1**, p. 78]. Our work will be simplified by the following lemma:

LEMMA. *If $B \subseteq \boldsymbol{R}$ and f is a continuous injection of B into $\boldsymbol{R}$, then f^{-1} is continuous at the image of any interior point of B.*

Proof. Let $y = f(x)$ be such a point. There is a closed interval I contained in B and centered at x. The image $f(I)$ is a connected, compact subset of $\boldsymbol{R}$, hence a closed interval. The injectivity of f forces y to lie in its interior. Any sequence converging to y must eventually lie in $f(I)$ and its preimages in I. Thus the problem restricts to the compact case, and f^{-1} is continuous at y. Q.E.D.

The lemma shows that we need only check continuity at the images of points in B^*. Open sets are stable since every point is interior.

Now we turn to the fourth class listed in the main theorem. *If a stable set B is not open, compact, or an interval, and if x_0 is in B^*, then x_0 must be screened.* The proof

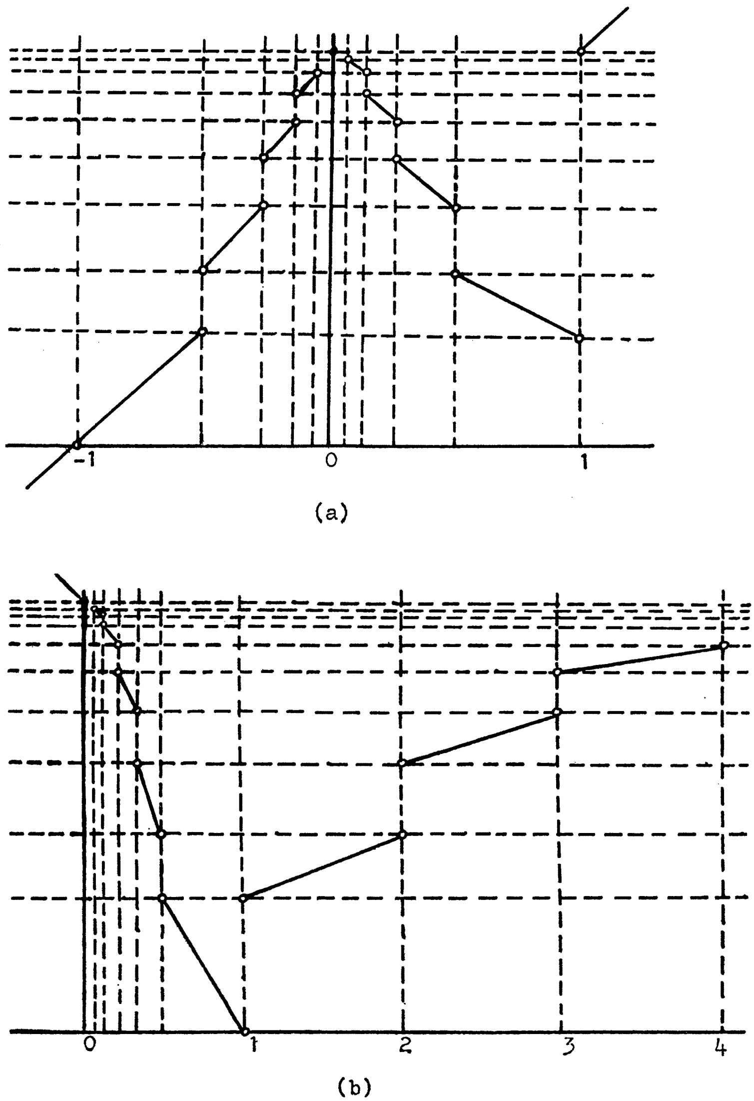


FIG. 1. (a) and (b)

that B contains an interval I on one side of x_0 is modeled on example 5. If there is no such interval, then there are deleted points converging to x_0 from both sides. Map the parts of B in the segments between these points into intervals between 0 and 1 in an alternating pattern as in example 5. There is a missing limit point somewhere (perhaps at infinity) say x_1. Extend f either linearly as in example 5 or using something like a translation of arctangent if x_1 must be taken at infinity as in example 2, so that there is a sequence in B converging to x_1 whose images converge to 1. Now put $f(x_0) = 1$. The inverse is discontinuous at 1 contradicting the assumption that B is stable. Thus B must contain an interval I on one side of x_0, and we may assume that $I = \text{co}(x_0)$.

We construct case-by-case examples to show that B contains points arbitrarily near x_0 on both sides. If not, we may assume that I lies to the right of x_0 and that B deletes an interval J to the left of x_0. Let $B' = B \cap \{x; x < x_0\}$.

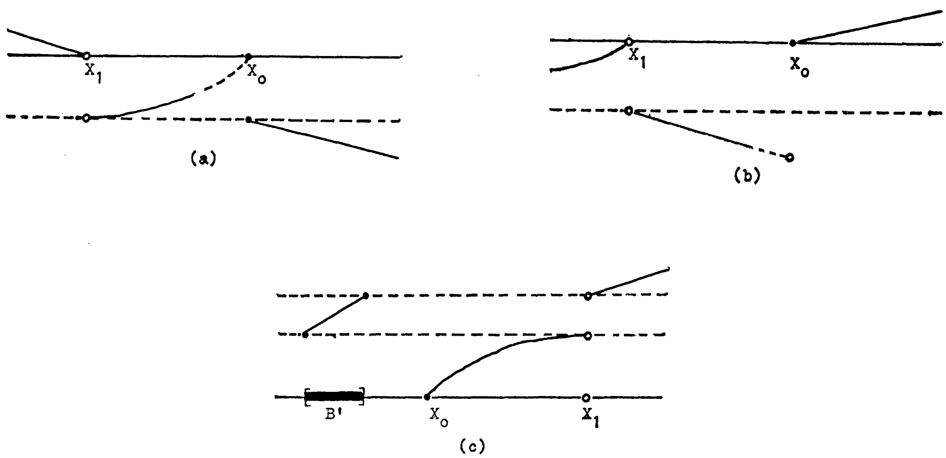


FIG. 2. (a), (b), and (c)

Case 1. If B' is not compact then it has a missing limit point x_1. This is not x_0 because of the deleted interval J. It might be $-\infty$. If B approaches x_1 from above, define f by arctan $(x - x_0)$ between x_1 and x_0 and linearly outside this region as in Figure 2a. If B approaches x_1 only from below, define f by arctan $(x - x_1)$ below x_1 and linearly in the remaining regions as in Figure 2b. In each case f is continuous since B misses x_1 and J, but the inverse is discontinuous at $f(x_0)$.

Case 2. If B' is compact and not empty, then it includes its extreme points. There is a limit point x_1 missing from B to the right of x_0, perhaps at $+\infty$. Define f by arctan $(x - x_0)$ between x_0 and x_1 and linearly in the remaining regions as in Figure 2c. Again f^{-1} is discontinuous at the image either of inf(B') or of sup(B').

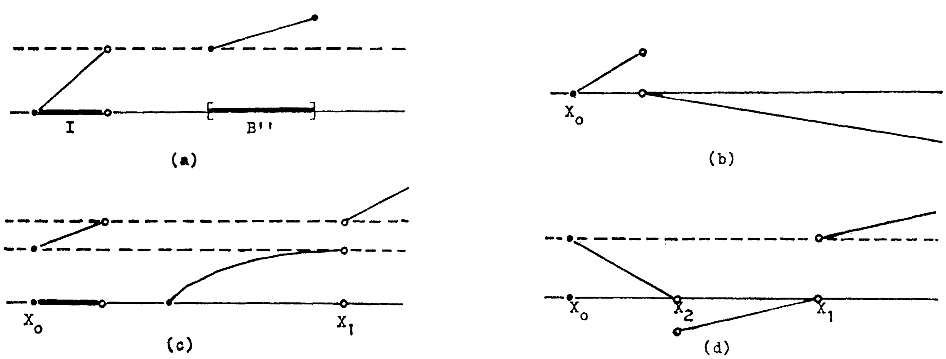


FIG. 3. (a), (b), (c) and (d)

Case 3. If B' is empty, then I is bounded since B is not an interval. Put $B'' = B\backslash I$. If B'' is compact, then I is not closed and inf $B'' \neq$ sup I since in these cases B would be compact. Define f as in Figure 3a, taking advantage of the gap between I and B'' to create a discontinuity in f^{-1}. If sup(I) were the only missing limit point of B'', it could not lie in B since then B would be compact. Figure 3b describes the counterexample. Now suppose all missing limit points of B'' are approached in B'' only from below, and let x_1 be such a point. (It might lie at $+\infty$.) If $\sup(I)$ were in B, it would be in I. Thus it is not in B''. But $\inf(B'')$ is in B'', since missing limit points are approached only from below. Thus $\sup(I) \neq \inf(B'')$, and there is a gap between I and B''. Define f by arctangent between $\inf(B'')$ and x_1, linearly on I starting with $f(x_0) = \text{arctangent}\,(x_1)$, and linearly above x_1 if necessary. See Figure 3c. In the only remaining case there is a missing limit point x_1 of B'' between $\sup(I)$ and $+\infty$ and approached in B'' from above. Since x_1 is not $\sup(I)$, there is a point x_2 between x_0 and x_1 and not in B. Define f linearly on all regions as in Figure 3d. The inverse is discontinuous at $f(x_0)$. This eliminates the last case and shows that x_0 must be screened.

What is going on is the following. The lemma shows that continuity of the inverse can only fail at points in the image of B^*. If $y = f(x)$ for x in B^*, continuity at y can be broken if images of components of B distant from x can come close to y. We must insulate x against such behavior. If x is screened, the image of the interval I is an interval on one side of y, and a sequence of points of B approaching x must have images approaching y from the other side. This is not quite enough to do the job. There can be no sequence of components of B, distant from x, which can be mapped into the gaps between these images. The problem is illustrated by example 6. Requiring finitely many components would be far too strong a condition as the images of many components may be tied down by proximity to a boundary point. What is needed is that in some sense there are only finitely many "free" components whose images may be placed anywhere.

To show that the classes of the theorem include all stable sets it remains to show that *a set B which is not compact, open, or an interval and such that every point in B^* is screened but which is not *-finite is not stable.* Since boundary points are screened, the components of B are intervals. Let $x_0 \in B^*$, let $I = \text{co}(x_0)$, and let $\{b_n\}$ be a sequence of points in B converging monotonically to x_0 from the other side. Put $f(x_0) = 1$, map I linearly onto an interval above 1, and $\text{co}(b_n)$ into the interval

$$\left(1 - \frac{1}{3n-2}, 1 - \frac{1}{3n-1}\right).$$

We construct "free portions" of B as follows. The components of B are either open intervals or $\text{co}(x_a)$ for $x_a \in B^*$. Since B is not *-finite, there are neighborhoods of each x_a such that the remainder of B still has infinitely many components after they are deleted. There are still infinitely many components remaining if the deleted neighborhoods are shrunk to lie in intervals B_a of length less than $\frac{1}{4}$ the length of $\text{co}(x_a)$ and with one end not in B. Let B_0 be the interval so associated with x_0. "Free portions" of B are constructed by taking

$$C_a = B \cap (\text{co}(x_a) \cup B_a)$$

or $C_a = B \cap (\text{co}(x_a) \cup B_a \cup B_b)$ if $\text{co}(x_a)$ should happen to be a closed interval with x_b at its other end. Open components outside any of the C_a so constructed remain unchanged as "free portions." The lack of *-finiteness and choice of the B_a shows that infinitely many of the "free portions" are distinct and nonoverlapping. Since these have been carefully constructed to be separated by points not in B, we may define f independently on them. Since all but finitely many of the co(b_n) fall inside B_0, f has yet to be defined on infinitely many "free portions" all lying outside B_0. Mapping at least one of these into each of the intervals

$$\left(1 - \frac{1}{3n-1}, 1 - \frac{1}{3n}\right)$$

will make f^{-1} discontinuous at 1. The definition of f is complete except possibly for some components lying inside B_0. These may be mapped into the intervals

$$\left(1 - \frac{1}{3n}, 1 - \frac{1}{3n+1}\right)$$

in such a way that f is continuous.

The last part of the theorem to be proved is that *if B is a *-finite set such that every point of B^* is screened, then B is stable.* Suppose f is a continuous injection of B into $\boldsymbol{R}$. By the lemma we need only check continuity of f^{-1} at $y_0 = f(x_0)$ for $x_0 \in B^*$. Since x_0 is screened, B contains an interval $I = \text{co}(x_0)$ on one side of x_0 and a sequence $\{b_n\}$ converging monotonically to x_0 on the other. If f^{-1} were discontinuous at y_0 there would be a sequence $\{y_n\} \subset f(B)$ converging to y_0 such that the points $x_n = f^{-1}(y_n)$ fail to converge to x_0. By passing to a subsequence we may assume that x_0 is not a cluster point of the sequence $\{x_n\}$. Since f^{-1} is continuous when restricted to $f(I)$, we may assume that $\{y_n\} \cap f(I) = \varnothing$ and $\{x_n\} \cap I = \varnothing$. Since the b_n converge to x_0 and can be taken separated by points not in B, there is an $N > 0$ such that

$$|b_m - x_0| < |x_n - x_0| \qquad \text{for every } n$$

and

$$b_m \notin \text{co}(x_n) \qquad \text{for any } n$$

whenever $m > N$.

We know co(x_n) and co(x_m) are either disjoint or identical. Suppose they are identical for a subsequence, $\text{co}(x_{n_k}) = J$ for $k = 1, 2, 3, \cdots$. Then $f(J)$ is an interval, and $f(y_{n_k}) \in f(J)$. Since the y_{n_k} converge to y_0, this forces $f(J)$ to have y_0 as an endpoint. Thus $f(I) \cup f(J)$ is an interval with y_0 in its interior. But this cannot happen since the $f(b_n)$ converge to y_0 and $f(b_n) \notin f(I) \cup f(J)$ for $n > N$ by the injectivity of f. Thus we may assume that $\text{co}(x_n) \cap \text{co}(x_m) = \varnothing$ for $n \neq m$ by passing to a subsequence if necessary. Since all points of B^* are screened, all components of B are nontrivial intervals. By the second characterization of *-finiteness the x_n must cluster somewhere in B. Say there is a subsequence $\{x_{n_i}\}$ with $\lim_{i\to\infty} x_{n_i} = b$. Since f is continuous, $\lim_{i\to\infty} f(x_{n_i}) = f(b)$. But $f(x_{n_i}) = y_{n_i}$, and $\lim_{i\to\infty} y_{n_i} = y_0$. Thus $f(b) = y_0 = f(x_0)$. Since $b \neq x_0$, this contradicts the injectivity of f. Thus f^{-1} must have been continuous at y_0. Since f was an arbitrary continuous injection of B into $\boldsymbol{R}$, B is stable and the proof is complete.

The author was supported by a National Science Foundation Graduate Fellowship.

References

1. J. Dieudonné, Foundations of Modern Analysis, Academic Press, New York, 1960.
2. J. L. Kelley, General Topology, Van Nostrand, Princeton, 1955.

CIRCULANT MATRICES AND ALGEBRAIC EQUATIONS

ROGER CHALKLEY, University of Cincinnati

1. Introduction. For each monic polynomial

$$f(X) = X^n + c_1X^{n-1} + \cdots + c_n \tag{1}$$

of degree $n \geqq 1$ over the field C of complex numbers, there exist elements $a_1, \cdots, a_n$ in C such that the $n \times n$ circulant matrix

$$A = \begin{bmatrix} a_1 & a_2 & a_3 & \cdots & a_n \\ a_n & a_1 & a_2 & \cdots & a_{n-1} \\ a_{n-1} & a_n & a_1 & \cdots & a_{n-2} \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ a_2 & a_3 & a_4 & \cdots & a_1 \end{bmatrix} \tag{2}$$

has $f(X)$ as its characteristic polynomial in the sense

$$f(X) = \det(XI_n - A). \tag{3}$$

In Section 2, we prove the preceding statement and the identity

$$\det(XI_n - A) = \prod_{s=1}^{n} \left(X - \sum_{k=1}^{n} a_k\zeta_n^{(k-1)(s-1)}\right), \tag{4}$$

where ζ_n denotes a primitive nth root of unity. Thus, the eigenvalues

$$\xi_s = \sum_{k=1}^{n} a_k\zeta_n^{(k-1)(s-1)}, \text{ for } s = 1, 2, \cdots, n, \tag{5}$$

of A are the roots of $f(X)$ in C. *The problem to solve $f(X) = 0$ in C can therefore be replaced by the problem to find an $n \times n$ circulant matrix over C which has $f(X)$ as its characteristic polynomial.* For $n = 1, 2, 3, 4$, all such $n \times n$ circulant matrices are presented in Section 4. There are $n!$ corresponding sets of formulas for the roots of $f(X)$.

2. Properties of circulant matrices. Let n be a positive integer and let F be a

The author was supported by a National Science Foundation Graduate Fellowship.

References

1. J. Dieudonné, Foundations of Modern Analysis, Academic Press, New York, 1960.
2. J. L. Kelley, General Topology, Van Nostrand, Princeton, 1955.

CIRCULANT MATRICES AND ALGEBRAIC EQUATIONS

ROGER CHALKLEY, University of Cincinnati

1. Introduction. For each monic polynomial

$$f(X) = X^n + c_1X^{n-1} + \cdots + c_n \tag{1}$$

of degree $n \geqq 1$ over the field C of complex numbers, there exist elements $a_1, \cdots, a_n$ in C such that the $n \times n$ circulant matrix

$$A = \begin{bmatrix} a_1 & a_2 & a_3 & \cdots & a_n \\ a_n & a_1 & a_2 & \cdots & a_{n-1} \\ a_{n-1} & a_n & a_1 & \cdots & a_{n-2} \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ a_2 & a_3 & a_4 & \cdots & a_1 \end{bmatrix} \tag{2}$$

has $f(X)$ as its characteristic polynomial in the sense

$$f(X) = \det(XI_n - A). \tag{3}$$

In Section 2, we prove the preceding statement and the identity

$$\det(XI_n - A) = \prod_{s=1}^{n} \left(X - \sum_{k=1}^{n} a_k\zeta_n^{(k-1)(s-1)}\right), \tag{4}$$

where ζ_n denotes a primitive nth root of unity. Thus, the eigenvalues

$$\xi_s = \sum_{k=1}^{n} a_k\zeta_n^{(k-1)(s-1)}, \text{ for } s = 1, 2, \cdots, n, \tag{5}$$

of A are the roots of $f(X)$ in C. *The problem to solve $f(X) = 0$ in C can therefore be replaced by the problem to find an $n \times n$ circulant matrix over C which has $f(X)$ as its characteristic polynomial.* For $n = 1, 2, 3, 4$, all such $n \times n$ circulant matrices are presented in Section 4. There are $n!$ corresponding sets of formulas for the roots of $f(X)$.

2. Properties of circulant matrices. Let n be a positive integer and let F be a field which contains a primitive nth root ζ_n of unity. In particular, the characteristic of F cannot divide n. For $r = 1,\cdots,n$ and $s = 1,\cdots,n$, set $\delta_{rs} = 1$ when $r = s$; and, $\delta_{rs} = 0$ when $r \neq s$.

LEMMA. *We have*

$$\frac{1}{n}\sum_{j=1}^{n}\zeta_n^{(j-1)(s-r)} = \delta_{rs}, \text{ for } r = 1,\cdots,n \text{ and } s = 1,\cdots,n. \tag{6}$$

Proof. Set $\rho = \zeta_n^{(s-r)}$. For $r \neq s$, we obtain $\rho \neq 1$, $\rho^n = 1$, and

$$\frac{1}{n}(1+\rho+\rho^2+\cdots+\rho^{n-1}) = \frac{1-\rho^n}{n(1-\rho)} = 0.$$

If $r = s$, then $\rho = 1$ and $(1/n)(n) = 1$. This completes the proof.

Let L_n and M_n be the $n \times n$ matrices whose components of row index r and column index s are

$$\lambda_{rs} = \frac{1}{n}\zeta_n^{-(r-1)(s-1)} \text{ and } \mu_{rs} = \zeta_n^{(r-1)(s-1)}.$$

The element in the (r,s) position of L_nM_n is

$$\sum_{j=1}^{n}\lambda_{rj}\mu_{js} = \frac{1}{n}\sum_{j=1}^{n}\zeta_n^{(j-1)(s-r)} = \delta_{rs}.$$

Thus, we have $L_nM_n = I_n$ and $M_n^{-1} = L_n$.

THEOREM. *Suppose $n \times n$ matrices A and D over F satisfy $AM_n = M_nD$. Then, A is a circulant matrix if and only if D is a diagonal matrix.*

Proof. Let a_{rs} and d_{rs} be the components of row index r and column index s for A and D.

(i) Suppose D is a diagonal matrix. With $d_{rs} = d_{rr}\delta_{rs}$ and $A = M_nDM_n^{-1}$, we obtain

$$a_{rs} = \sum_{j=1}^{n}\sum_{k=1}^{n}\mu_{rj}d_{jj}\delta_{jk}\lambda_{ks} = \sum_{j=1}^{n}\mu_{rj}d_{jj}\lambda_{js} = \frac{1}{n}\sum_{j=1}^{n}d_{jj}\zeta_n^{(j-1)(r-s)}.$$

Since the elements in the (r,s) and (r',s') positions of A are equal when $r-s \equiv r'-s'$ $(\text{mod}\, n)$, A is a circulant matrix.

(ii) Suppose A is a circulant matrix. For $r = 1,\cdots,n$ and $s = 1,\cdots,n$, set $a_{r,s+n} = a_{rs}$. With $D = M_n^{-1}AM_n$, we find

$$\begin{aligned} d_{rs} &= \sum_{j=1}^{n}\lambda_{rj}\sum_{k=1}^{n}a_{jk}\zeta_n^{(k-1)(s-1)} = \sum_{j=1}^{n}\lambda_{rj}\sum_{k=j}^{n+j-1}a_{jk}\zeta_n^{(k-1)(s-1)} \\ &= \frac{1}{n}\sum_{j=1}^{n}\zeta_n^{-(r-1)(j-1)}\sum_{k=1}^{n}a_{j,j-1+k}\zeta_n^{(j-1+k-1)(s-1)} \\ &= \frac{1}{n}\sum_{j=1}^{n}\zeta_n^{(j-1)(s-r)}\sum_{k=1}^{n}a_{1k}\zeta_n^{(k-1)(s-1)}. \end{aligned} \tag{7}$$

By (6) and (7), we have $d_{rs} = 0$ when $r \neq s$. This completes the proof.

COROLLARY 1. *The $n \times n$ circulant matrices over F form a commutative ring under matrix addition and multiplication which is isomorphic to the ring of $n \times n$ diagonal matrices over F.*

Proof. By the theorem, we obtain a one-to-one correspondence between the two sets of matrices which preserves the operations of matrix addition and multiplication. Since the $n \times n$ diagonal matrices over F form a commutative ring, the same is true for the $n \times n$ circulant matrices over F.

COROLLARY 2. *An $n \times n$ matrix A over F is circulant if and only if all the column vectors of M_n are eigenvectors of A.*

Proof. Set $D = M_n^{-1}AM_n$. From $AM_n = M_nD$, we note that D is a diagonal matrix if and only if each column vector of M_n is an eigenvector of A. To complete the proof, we use the theorem.

COROLLARY 3. *Let an $n \times n$ circulant matrix A be defined over F by* (2), *and let $\xi_1, \xi_2, \cdots, \xi_n$ be defined by* (5). *Then* $\det A = \xi_1\xi_2 \cdots \xi_n$.

Proof. For the diagonal matrix $D = M_n^{-1}AM_n$, we use (7) and (6) with $r = s$ and $a_{1k} = a_k$ to obtain $d_{ss} = \xi_s$. This yields

$$\det A = \det(M_n^{-1}AM_n) = \det D = \prod_{s=1}^{n} \xi_s.$$

For other proofs of this well-known result, see [**2**] or [**1**].

COROLLARY 4. *If A is defined over F by* (2), *then* (4) *is valid.*

Proof. With the notation used for Corollary 3, we obtain

$$\det(XI_n - A) = \det(M_n^{-1}(XI_n - A)M_n) = \det(XI_n - D) = \prod_{s=1}^{n} (X - \xi_s).$$

This completes the proof.

By definition, an $n \times n$ permutation matrix is a matrix obtained from I_n by a permutation of rows (or columns). There are $n!$ such matrices.

COROLLARY 5. *Suppose $[a_1, \cdots, a_n]$ and $[b_1, \cdots, b_n]$ are the first rows of $n \times n$ circulant matrices A and B over F. Then, A and B have the same characteristic polynomial if and only if there exists an $n \times n$ permutation matrix P such that*

$$[b_1, \cdots, b_n] = [a_1, \cdots, a_n](M_nPM_n^{-1}). \tag{8}$$

Proof. Define $\xi_1, \cdots, \xi_n$ and $\eta_1, \cdots, \eta_n$ by

$$[\xi_1, \cdots, \xi_n] = [a_1, \cdots, a_n]M_n \text{ and } [\eta_1, \cdots, \eta_n] = [b_1, \cdots, b_n]M_n.$$

We use Corollary 4 to obtain

$$\det(XI_n - A) = \prod_{s=1}^{n} (X - \xi_s) \text{ and } \det(XI_n - B) = \prod_{s=1}^{n} (X - \eta_s).$$

Thus, the characteristic polynomials of A and B are equal if and only if there exists an $n \times n$ permutation matrix P such that

$$(9) \qquad [\eta_1, \cdots, \eta_n] = [\xi_1, \cdots, \xi_n]P.$$

To complete the proof, we rewrite (9) in the form (8).

COROLLARY 6. *Suppose A is an $n \times n$ circulant matrix over F. Then, an $n \times n$ matrix B over F is circulant with*

$$(10) \qquad \det(XI_n - B) = \det(XI_n - A)$$

if and only if there exists an $n \times n$ permutation matrix P such that

$$(11) \qquad B = (M_n P^T M_n^{-1}) A (M_n P M_n^{-1}).$$

Proof. (i) Suppose B is circulant and related to A by (10). Then, the characteristic polynomials of the diagonal matrices $D = M_n^{-1}AM_n$ and $E = M_n^{-1}BM_n$ are equal. Hence, there exists an $n \times n$ permutation matrix P such that $E = P^TDP$. This yields (11).

(ii) Suppose (11) is satisfied. Then, we easily obtain (10). Moreover, $M_n^{-1}AM_n$ is diagonal; $P^T(M_n^{-1}AM_n)P$ is diagonal; and, with (11), B is circulant. This completes the proof.

Set $R_n = (1/n)M_n^2$. The element in the (r, s) position of R_n is

$$\frac{1}{n} \sum_{j=1}^{n} \mu_{rj}\mu_{js} = \frac{1}{n} \sum_{j=1}^{n} \zeta_n^{(j-1)(r+s-2)};$$

it equals 1 when n divides $r + s - 2$ and it equals 0 otherwise. Thus, R_n is a symmetric permutation matrix. To obtain R_n from I_n, we can interchange the jth and $(n + 2-j)$th rows of I_n for $j = 2, 3, \cdots, n$. Also, R_n results when the jth and $(n + 2-j)$th columns of I_n are interchanged for $j = 2, 3, \cdots, n$. With $R_n^2 = I_n$, we have $M_n^4 = n^2I_n$ and $M_n^{-1} = (1/n^2)M_n^3$.

Suppose A is given by (2). We can obtain the transpose A^T of A as follows: for $j = 2, 3, \cdots, n$, interchange the jth and $(n+2-j)$th rows of A; then, for $j = 2, 3, \cdots, n$, interchange the jth and $(n+2-j)$th columns of the resulting matrix. Thus, we have $R_nAR_n = A^T$.

Suppose $n \times n$ matrices P and Q over F satisfy $Q = R_nPR_n$. We note that Q is circulant if and only if P is circulant; Q is diagonal if and only if P is diagonal; and, Q is a permutation matrix if and only if P is a permutation matrix. The relations

$$M_nPM_n^{-1} = M_n^{-1}QM_n \text{ and } M_nP^TM_n^{-1} = M_n^{-1}Q^TM_n$$

can be used to reformulate (8) and (11).

COROLLARY 7. *Let A and D be $n \times n$ matrices over F such that $AM_n = M_nD$. Then, one of A and D is circulant if and only if the other is diagonal.*

Proof. We have $M_nAM_n^{-1} = M_n^2DM_n^{-2} = R_nDR_n$. By the theorem, A is diagonal if and only if $M_nAM_n^{-1}$ is circulant; but, R_nDR_n is circulant if and only if D is circulant. Directly from the theorem, A is circulant if and only if D is diagonal. This completes the proof.

Henceforth, we specialize F to be the field C of complex numbers.

COROLLARY 8. *If $f(X)$ is a monic polynomial of degree $n \geqq 1$ over C, then $f(X)$ is the characteristic polynomial of some $n \times n$ circulant matrix over C.*

Proof. There exist elements $\eta_1, \eta_2, \cdots, \eta_n$ in C such that

$$f(X) = (X - \eta_1)(X - \eta_2)\cdots(X - \eta_n).$$

Let D be the $n \times n$ diagonal matrix with $d_{ss} = \eta_s$. Set $A = M_nDM_n^{-1}$. Then, A is an $n \times n$ circulant matrix over C and

$$\det(XI_n - A) = \det(M_n^{-1}(XI_n - A)M_n) = \det(XI_n - D) = f(X).$$

This completes the proof.

Set $N_n = (1/\sqrt{n})M_n$. Since the complex conjugate of ζ_n^k is ζ_n^{-k}, the conjugate $\bar{N}_n$ of N_n satisfies $\bar{N}_n = N_n^{-1}$. With $N_n^T = N_n$ and $\bar{N}_n^T = N_n^{-1}$, the matrix N_n is both symmetric and unitary. We note that $N_n^2 = R_n$ and $N_n^{-1} = N_n^3$. When $n \geqq 3$, N_n generates a cyclic group of order 4. For each $n \times n$ matrix P, we have $M_nPM_n^{-1} = N_nPN_n^{-1}$.

3. Several group representations. The elements of the symmetric group S_n are the permutations of n objects. When the objects are identified with the rows of I_n, we obtain an isomorphism of S_n onto the multiplicative group of $n \times n$ permutation matrices. As P ranges over the $n \times n$ permutation matrices, the corresponding matrices $M_nPM_n^{-1}$ also form a group under matrix multiplication which is isomorphic to S_n.

LEMMA. *The element in the $(1,1)$ position of $M_nPM_n^{-1}$ equals 1 and the elements in the other positions of the first row or first column equal 0.*

Proof. For $P = [\pi_{rs}]$ and $M_nPM_n^{-1} = [\alpha_{rs}]$, we find

$$\alpha_{rs} = \frac{1}{n}\sum_{j=1}^{n}\sum_{k=1}^{n}\zeta_n^{(r-1)(j-1)}\pi_{jk}\zeta_n^{-(k-1)(s-1)}$$

and

$$\alpha_{1s} = \frac{1}{n}\sum_{k=1}^{n}\zeta_n^{(k-1)(1-s)}\sum_{j=1}^{n}\pi_{jk} = \delta_{1s}.$$

Similarly, we obtain $\alpha_{r1} = \delta_{r1}$. This completes the proof.

PROPOSITION. *For $n \geqq 2$, let P range over all $n!$ of the permutation matrices of size $n \times n$. Then, the corresponding matrices of size $(n-1) \times (n-1)$ obtained by deletion of the first row and first column from each matrix $M_n P M_n^{-1}$ form a group under matrix multiplication which is isomorphic to S_n.*

Proof. This follows directly from the lemma.

Example 1. For $n = 3$, $\omega^2 + \omega + 1 = 0$, $\zeta_3 = \omega$,

$$M_3 = \begin{bmatrix} 1 & 1 & 1 \\ 1 & \omega & \omega^2 \\ 1 & \omega^2 & \omega \end{bmatrix}, \text{ and } 3M_3^{-1} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & \omega^2 & \omega \\ 1 & \omega & \omega^2 \end{bmatrix},$$

we follow the details of the proposition to obtain

$$(12) \quad \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} \omega & 0 \\ 0 & \omega^2 \end{bmatrix}, \begin{bmatrix} \omega^2 & 0 \\ 0 & \omega \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & \omega^2 \\ \omega & 0 \end{bmatrix}, \begin{bmatrix} 0 & \omega \\ \omega^2 & 0 \end{bmatrix}.$$

The six matrices of (12) form a group isomorphic to S_3.

The lemma and the proposition remain valid when M_n is replaced throughout by $G_n M_n$, where $G_n = [g_{rs}]$ is a nonsingular $n \times n$ matrix with

$$g_{r1} = g_{1r} = \delta_{r1}, \quad \text{for} \quad r = 1, 2, \cdots, n.$$

For instance, set

$$g_{rs} = -\zeta_n^{-(r-1)(s-1)}, \text{ for } r = 2, 3, \cdots, n \text{ and } s = 2, 3, \cdots, n,$$

$H_n = G_n M_n$, and $H_n = [h_{rs}]$. We find $h_{rr} = 1 - n$ for $r = 2, 3, \cdots, n$ and $h_{rs} = 1$ otherwise; moreover,

$$nH_n^{-1} = \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & -1 & 0 & \cdots & 0 \\ 1 & 0 & -1 & \cdots & 0 \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ 1 & 0 & 0 & \cdots & -1 \end{bmatrix}$$

Example 2. For $n = 3$, the matrices $H_3 P H_3^{-1}$ yield

$$(13) \quad \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} -1 & -1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ -1 & -1 \end{bmatrix}, \begin{bmatrix} -1 & -1 \\ 0 & 1 \end{bmatrix}.$$

The six matrices of (13) form a group which is isomorphic to S_3. For related details, see [3].

The even permutations in S_n correspond to matrices P, $M_n P M_n^{-1}$, $H_n P H_n^{-1}$, etc., with determinant equal to 1; and, the odd permutations in S_n correspond to matrices with determinant equal to -1. For example, the alternating subgroup of S_3 is represented by the first three matrices in (13).

4. The roots of $f(X)$, for $n = 1,2,3,4$. First, we relate $c_1,\cdots,c_n$ to $a_1,\cdots,a_n$ in (1), (2), and (3). From (1), the $(n-k)$th derivative $f^{(n-k)}(X)$ of $f(X)$ yields

$$c_k = \frac{f^{(n-k)}(0)}{(n-k)!}, \quad \text{for } k = 1,2,\cdots,n.$$

By application to (3) of rules for the differentiation of a determinant, we conclude: *c_k equals the sum of the determinants of the principal $k \times k$ submatrices of $-A$.*

When $n = 1$, we find $\zeta_1 = 1$, $c_1 = -a_1$, and $\xi_1 = a_1 = -c_1$.

For $n = 2$, we need $\zeta_2 = -1$, $c_1 = -2a_1$, $c_2 = a_1^2 - a_2^2$, $a_1 = -c_1/2$, and a_2 such that $a_2^2 = (c_1^2/4) - c_2$; by (5), the roots of $f(X)$ in C are $a_1 + a_2$ and $a_1 - a_2$.

In each case, we have $c_1 = -na_1$. For $n = 3$ and $n = 4$, it is convenient to make a preliminary transformation so that $c_1 = 0$.

A 3×3 circulant matrix A given by (2) with $n = 3$ has

$$\det(XI_3 - A) = X^3 + c_2X + c_3 \tag{14}$$

as its characteristic polynomial if and only if its components satisfy

$$-3a_1 = 0, \quad -3a_2a_3 = c_2, \text{ and } -a_2^3 - a_3^3 = c_3. \tag{15}$$

When $c_2 = c_3 = 0$, set $y_0 = z_0 = 0$; otherwise, $T^2 + c_3T - (c_2/3)^3$ has a nonzero root t_0, let y_0 satisfy $Y^3 = t_0$, and set $z_0 = (-c_2)/(3y_0)$. In this way, (15) is satisfied with $a_1 = 0$, $a_2 = y_0$, $a_3 = z_0$. We use (8) and the six matrices $M_3PM_3^{-1}$ upon which (12) was based to obtain the first rows

$$\begin{gathered}[0, y_0, z_0],\ [0, y_0\omega, z_0\omega^2],\ [0, y_0\omega^2, z_0\omega], \\ [0, z_0, y_0],\ [0, z_0\omega, y_0\omega^2],\ [0, z_0\omega^2, y_0\omega]\end{gathered} \tag{16}$$

of all 3×3 circulant matrices (2) for (14). Now, we can use (5) to write all six forms of Cardan's formulas for the roots of a cubic. Namely, for each selection of $[a_1, a_2, a_3]$ from (16), the corresponding elements

$$a_2\omega^{(s-1)} + a_3\omega^{2(s-1)}, \text{ for } s = 1,2,3,$$

are the roots in C of $X^3 + c_2X + c_3$.

A 4×4 circulant matrix A given by (2) with $n = 4$ has

$$\det(XI_4 - A) = X^4 + c_2X^2 + c_3X + c_4 \tag{17}$$

as its characteristic polynomial if and only if its components satisfy

$$\begin{gathered}-4a_1 = 0, \quad -4a_2a_4 - 2a_3^2 = c_2, \quad -4a_2^2a_3 - 4a_3a_4^2 = c_3, \text{ and} \\ -a_2^4 + a_3^4 - a_4^4 + 2a_2^2a_4^2 - 4a_2a_3a_4^2 = c_4.\end{gathered} \tag{18}$$

When $c_2 = c_3 = c_4 = 0$, set $u_0 = v_0 = w_0 = 0$; otherwise, the equation

$$(4V^2)^3 + 2c_2(4V^2)^2 + (c_2^2 - 4c_4)(4V^2) - c_3^2 = 0$$

has a nonzero solution v_0 and we select u_0, w_0 to satisfy

$$UW = -\frac{v_0^2}{2} - \frac{c_2}{4} \text{ and } U^2 + W^2 = -\frac{c_3}{4v_0}.$$

We can verify that $a_1 = 0, a_2 = u_0, a_3 = v_0, a_4 = w_0$ is a solution of (18); similar details were given in [1]. Next, we apply (8). As P ranges over the permutation matrices of size 4×4, the formula

$$[a_1, a_2, a_3, a_4] = [0, u_0, v_0, w_0](M_4PM_4^{-1})$$

specifies the first row $[a_1, a_2, a_3, a_4]$ of each 4×4 circulant matrix A for (17). With $i^2 = -1$ and $\zeta_4 = i$, we use (5) to conclude that the corresponding elements

$$a_2 i^{(s-1)} + a_3 i^{2(s-1)} + a_4 i^{3(s-1)}, \quad \text{for } s = 1, 2, 3, 4,$$

are the roots in C of $X^4 + c_2X^2 + c_3X + c_4$. Thus, there are $4! = 24$ sets of solution formulas for a biquadratic analogous to the $3! = 6$ forms of Cardan's formulas for a cubic.

References

1. R. Chalkley, Cardan's formulas and biquadratic equations, this MAGAZINE, 47 (1974) 8–14.
2. T. Muir, A Treatise on the Theory of Determinants, Dover, New York, 1960, pp. 442–445.
3. E. D. Nering, Linear Algebra and Matrix Theory, Wiley, New York, 1970, pp. 306–308.

AN INVARIANT RELATION IN CHAINS OF TANGENT CIRCLES

YOSHIMASA MICHIWAKI, Gunma University, MAKOTO ŌYAMA, Gunma Prefectural Ōmama Senior High School and TOSHIO HAMADA, Gunma Ken Nitta Chō Ritsu Kisaki Lower Secondary School

1. Preliminaries. Interest in relations among the radii of tangent circles has a long history. In the western world the names of Apollonius, Archimedes, Steiner and Casey among others are associated with problems and theorems involving such systems of circles. In the orient this area of investigation has had a singular fascination. In particular it had an active and extensive development in Japan during the so-called *Wasan* period. [The word *Wasan* is the Japanese word meaning the development of mathematics in Japan and refers to the period prior to about 1850. Extensive references can be found in [6].]

In this paper we derive an interesting, and we hope new, invariant relation among the circles of two Steiner chains. As a corollary we obtain a generalization of a classic result of S. Kenmochi. The methods are entirely elementary.

When $c_2 = c_3 = c_4 = 0$, set $u_0 = v_0 = w_0 = 0$; otherwise, the equation

$$(4V^2)^3 + 2c_2(4V^2)^2 + (c_2^2 - 4c_4)(4V^2) - c_3^2 = 0$$

has a nonzero solution v_0 and we select u_0, w_0 to satisfy

$$UW = -\frac{v_0^2}{2} - \frac{c_2}{4} \text{ and } U^2 + W^2 = -\frac{c_3}{4v_0}.$$

We can verify that $a_1 = 0, a_2 = u_0, a_3 = v_0, a_4 = w_0$ is a solution of (18); similar details were given in [1]. Next, we apply (8). As P ranges over the permutation matrices of size 4×4, the formula

$$[a_1, a_2, a_3, a_4] = [0, u_0, v_0, w_0](M_4 P M_4^{-1})$$

specifies the first row $[a_1, a_2, a_3, a_4]$ of each 4×4 circulant matrix A for (17). With $i^2 = -1$ and $\zeta_4 = i$, we use (5) to conclude that the corresponding elements

$$a_2 i^{(s-1)} + a_3 i^{2(s-1)} + a_4 i^{3(s-1)}, \quad \text{for } s = 1, 2, 3, 4,$$

are the roots in C of $X^4 + c_2X^2 + c_3X + c_4$. Thus, there are $4! = 24$ sets of solution formulas for a biquadratic analogous to the $3! = 6$ forms of Cardan's formulas for a cubic.

References

1. R. Chalkley, Cardan's formulas and biquadratic equations, this MAGAZINE, 47 (1974) 8–14.
2. T. Muir, A Treatise on the Theory of Determinants, Dover, New York, 1960, pp. 442–445.
3. E. D. Nering, Linear Algebra and Matrix Theory, Wiley, New York, 1970, pp. 306–308.

AN INVARIANT RELATION IN CHAINS OF TANGENT CIRCLES

YOSHIMASA MICHIWAKI, Gunma University, MAKOTO ŌYAMA, Gunma Prefectural Ōmama Senior High School and TOSHIO HAMADA, Gunma Ken Nitta Chō Ritsu Kisaki Lower Secondary School

1. Preliminaries. Interest in relations among the radii of tangent circles has a long history. In the western world the names of Apollonius, Archimedes, Steiner and Casey among others are associated with problems and theorems involving such systems of circles. In the orient this area of investigation has had a singular fascination. In particular it had an active and extensive development in Japan during the so-called *Wasan* period. [The word *Wasan* is the Japanese word meaning the development of mathematics in Japan and refers to the period prior to about 1850. Extensive references can be found in [6].]

In this paper we derive an interesting, and we hope new, invariant relation among the circles of two Steiner chains. As a corollary we obtain a generalization of a classic result of S. Kenmochi. The methods are entirely elementary.

2. Definitions and lemmas.

DEFINITION 1. *A circle with center* O_i *and radius* r_i *will be referred to as the circle* O_i. *If* O_i *and* O_j *are circles with disjoint interiors,* t_{ij} *denotes the length of a common external tangent.*

DEFINITION 2. *If* O *and* O' *are circles of radii* R *and* r*, respectively, with* O' *interior to* O*, and if* $O_1, O_2, \cdots, O_m$ *is a sequence of circles such that* (a) O_i *is tangent to both* O *and* O', $i = 1, 2, \cdots, m$, *and* (b) O_i *and* O_{i+1} *are tangent,* $i = 1, 2, \cdots, m-1$, *then the sequence is called a Steiner chain of length* m *relative to* O *and* O'.

The following two lemmas are easily proved:

LEMMA 1. *If two circles* O_1 *and* O_2 *with disjoint interiors are tangent, then* $t_{12} = 2\sqrt{r_1 r_2}$.

LEMMA 2. *If* $ABCD$ *is an isosceles trapezoid with* $AB = CD$, *then* $BD^2 = BC \cdot AD + CD^2$.

LEMMA 3. *If* A, B, C, D *are four points in order on a circle* O, *then*

$$AC(AD \cdot CD + AB \cdot BC) - BD(AD \cdot AB + BC \cdot CD) = 0. \tag{1}$$

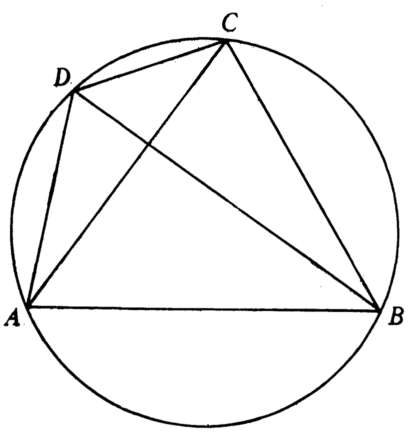


FIG. 1.

Proof.

$$\text{Area } \triangle ABC + \text{area } \triangle ACD = \text{area } \triangle ABD + \text{area } \triangle BCD.$$

But area $\triangle ABC = (AB \cdot BC \cdot AC)/4R$ with similar expressions for the other three triangles. Substitution and simplification lead to the relation given in the statement of the lemma.

3. The main theorem.

THEOREM 1. *If* $O_1, O_2, \cdots, O_m$ *and* $O_{m+1}, O_{m+2}, \cdots, O_{m+n}$ *are two Steiner chains* *relative to circles O and O′, each two circles in the system having disjoint interiors, then*

$$\frac{\sqrt{r_i r_j}}{|r_i - r_j|} t_{ij}$$

is a function of $i + j$, $1 \leqq i \leqq m < j \leqq m + n$. [4]

Proof. The general result will follow at once if we prove the following special case.

Let O_1, O_2, O_3, O_4 be four circles with disjoint interiors each tangent to both O and O' and suppose O_1 tangent to O_2 and O_3 tangent to O_4. Then

$$\frac{\sqrt{r_1 r_4}}{|r_1 - r_4|} t_{14} = \frac{\sqrt{r_2 r_3}}{|r_2 - r_3|} t_{23}.$$

Let $O_1 \cap O = A$, $O_2 \cap O = B$, $O_3 \cap O = C$, $O_4 \cap O = D$. Let I be on OB and H on OD such that $IO_4 \parallel HO_2 \parallel BD$. Then from similar triangles we have that $IO_4 = BD(R - r_4)/R$, $HO_2 = BD(R - r_2)/R$.

From Lemma 2, $IO_4 \cdot HO_2 + \overline{O_4H}^2 = \overline{O_2O_4}^2$.

Thus

$$BD = \frac{R}{\sqrt{R - r_2}\sqrt{R - r_4}} t_{24}. \tag{2}$$

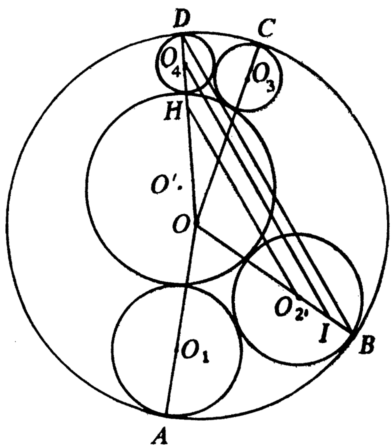


FIG. 2.

Similarly

$$AD = \frac{R}{\sqrt{R - r_1}\sqrt{R - r_4}} t_{14}, \tag{3}$$

$$AC = \frac{R}{\sqrt{R - r_1}\sqrt{R - r_3}} t_{13}, \tag{4}$$

$$CD = \frac{R}{\sqrt{R - r_3}\sqrt{R - r_4}} t_{34} = \frac{2\sqrt{r_3 r_4}}{\sqrt{R - r_3}\sqrt{R - r_4}} R, \tag{5}$$

$$BC = \frac{R}{\sqrt{R - r_2}\sqrt{R - r_3}}\, t_{23}, \tag{6}$$

$$AB = \frac{R}{\sqrt{R - r_1}\sqrt{R - r_2}}\, t_{12} = \frac{2\sqrt{r_1 r_2}}{\sqrt{R - r_1}\sqrt{R - r_2}}\, R. \tag{7}$$

Now using these relations in conjunction with Lemma 3, we have

$$AC(AD \cdot CD + AB \cdot BC) - BD(AD \cdot AB + BC \cdot CD) = 0.$$

$$\begin{aligned} & t_{13}\{\sqrt{r_3 r_4}(R - r_2)t_{14} + \sqrt{r_1 r_2}(R - r_4)t_{23}\} \\ & \quad - t_{24}\{\sqrt{r_1 r_2}(R - r_3)t_{14} + \sqrt{r_3 r_4}(R - r_1)t_{23}\} = 0. \end{aligned} \tag{8}$$

Completely analogous reasoning applied to the same four circles but with circle O' replacing circle O leads to

$$\begin{aligned} & t_{13}\{\sqrt{r_3 r_4}(r + r_2)t_{14} + \sqrt{r_1 r_2}(r + r_4)t_{23}\} \\ & \quad - t_{24}\{\sqrt{r_1 r_2}(r + r_3)t_{14} + \sqrt{r_3 r_4}(r + r_1)t_{23}\} = 0. \end{aligned} \tag{9}$$

$\{(8) + (9)\} \div (R + r)$ gives

$$t_{13}\{\sqrt{r_3 r_4}\, t_{14} + \sqrt{r_1 r_2}\, t_{23}\} - t_{24}\{\sqrt{r_1 r_2}\, t_{14} + \sqrt{r_3 r_4}\, t_{23}\} = 0. \tag{10}$$

$(9) - r \times (10)$ gives

$$t_{13}\{r_2\sqrt{r_3 r_4}\, t_{14} + r_4\sqrt{r_1 r_2}\, t_{23}\} - t_{24}\{r_3\sqrt{r_1 r_2}\, t_{14} + r_1\sqrt{r_3 r_4}\, t_{23}\} = 0. \tag{11}$$

From (10) and (11) we obtain

$$\begin{aligned} \frac{t_{13}}{t_{24}} &= \frac{\sqrt{r_1 r_2}\, t_{14} + \sqrt{r_3 r_4}\, t_{23}}{\sqrt{r_3 r_4}\, t_{14} + \sqrt{r_1 r_2}\, t_{23}} \\ &= \frac{r_3\sqrt{r_1 r_2}\, t_{14} + r_1\sqrt{r_3 r_4}\, t_{23}}{r_2\sqrt{r_3 r_4}\, t_{14} + r_4\sqrt{r_1 r_2}\, t_{23}} \end{aligned}$$

which, after some algebra, reduces to

$$\frac{\sqrt{r_1 r_4}}{|r_1 - r_4|}\, t_{14} = \frac{\sqrt{r_2 r_3}}{|r_2 - r_3|}\, t_{23}.$$

As observed at the outset, it now easily follows that

$$\frac{\sqrt{r_i r_j}}{|r_i - r_j|}\, t_{ij} = \frac{\sqrt{r_k r_l}}{|r_k - r_l|}\, t_{kl} \quad \text{if } i + j = k + l. \tag{12}$$

4. An application.

THEOREM 2. *If $O_1, O_2, \cdots, O_m$ and $O_{m+1}, O_{m+2}, \cdots, O_{2m}$ are two Steiner chains relative to circles O and O' of radii R and r, respectively, and if circles O_1, O_{2m} and*

O' are tangent to a line as are circles O_m, O_{m+1} and O', then

$$\frac{1}{\sqrt{r_1}} + \frac{1}{\sqrt{r_{m+1}}} = \frac{1}{\sqrt{r_m}} + \frac{1}{\sqrt{r_{2m}}}, \; m \geqq 2 \; [\mathbf{4}].$$

Proof. For notational simplicity we prove the theorem for the case $m = 4$. The proof of the theorem for general m follows the same lines.

From Theorem 1 we have the following relations:

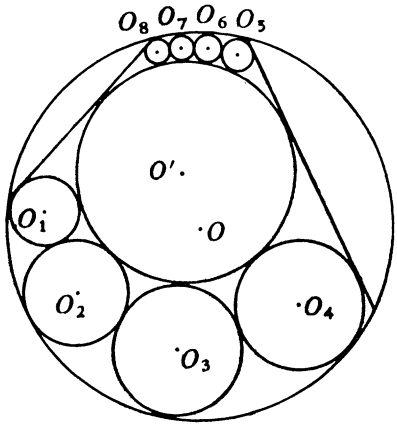


FIG. 3.

$$t_{36} = \frac{|r_6 - r_3|}{|r_5 - r_4|}\sqrt{\frac{r_5 r_4}{r_6 r_3}}\, t_{54},$$

$$t_{36} = \frac{|r_6 - r_3|}{|r_7 - r_2|}\sqrt{\frac{r_2 r_7}{r_6 r_3}}\, t_{27},$$

$$t_{27} = \frac{|r_7 - r_2|}{|r_8 - r_1|}\sqrt{\frac{r_8 r_1}{r_2 r_7}}\, t_{81},$$

(13) Thus $\displaystyle \frac{|r_7 - r_2|}{|r_5 - r_4|}\sqrt{\frac{r_5 r_4}{r_2 r_7}}\, t_{54} = \frac{|r_7 - r_2|}{|r_8 - r_1|}\sqrt{\frac{r_8 r_1}{r_2 r_7}}\, t_{81}.$

But since O_1, O_8 and O' are tangent to a line we have, using Lemma 1, that

$$t_{81} = 2\sqrt{r}(\sqrt{r_8} + \sqrt{r_1}).$$

Similarly $$t_{54} = 2\sqrt{r}(\sqrt{r_5} + \sqrt{r_4}).$$

Substitution in (13) leads to

$$\frac{1}{\sqrt{r_1}} + \frac{1}{\sqrt{r_5}} = \frac{1}{\sqrt{r_4}} + \frac{1}{\sqrt{r_8}}.$$

COROLLARY. *For $m = 2$, Theorem 2 gives*

$$\frac{1}{\sqrt{r_1}} + \frac{1}{\sqrt{r_3}} = \frac{1}{\sqrt{r_2}} + \frac{1}{\sqrt{r_4}}.$$

This is the theorem of S. Kenmochi [**1, 2, 4**] alluded to in the introduction.

5. Relation to Casey's theorem. If A, B, C, D are four points in order on a circle, then Ptolemy's theorem asserts that $AB \cdot CD + BC \cdot AD = AC \cdot BD$.

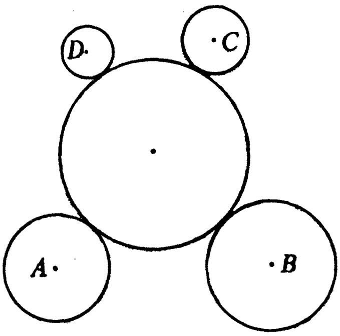


FIG. 4.

Casey observed that if A, B, C, D are the centers of four circles each of which is externally tangent to a fifth circle and if the four circles have pairwise disjoint interiors, then $t_{AB} \cdot t_{CD} + t_{BC} \cdot t_{AD} = t_{AC} \cdot t_{BD}$ where t_{AB} is the length of an external common tangent to circles A and B. This theorem of Casey has been extensively studied and versions of it are known in which the circles can be both externally and internally tangent though internal common tangents as well as external tangents must be considered. The equality in Ptolemy's theorem is both necessary and sufficient for the points A, B, C, D to be on a circle (including one of infinite radius, i.e., a line). Similarly, properly generalized, the equality in Casey's theorem [**3, 5**] is a necessary and sufficient condition for the four circles to be tangent to a fifth.

We have also noted that when A, B, C, D are on a circle we have the metric relation

$$AC(AD \cdot CD + AB \cdot BC) - BC(AD \cdot AB + BC \cdot CD) = 0.$$

Easy examples show that this condition is not a sufficient condition for the points A, B, C, D to be cyclic. However, it might still be wondered: if A, B, C, D are four circles with pairwise disjoint interiors each tangent to a fifth, will

$$t_{AC}(t_{AD} \cdot t_{CD} + t_{AB} \cdot t_{BC}) - t_{BD}(t_{AD} \cdot t_{AB} + t_{BC} \cdot t_{CD}) = 0?$$

The answer is: not in general. However, if the circles are also internally tangent to a sixth circle, then the relation follows as we now show.

THEOREM 3. *If O_1, O_2, O_3, O_4 are four circles with pairwise disjoint interiors each tangent to circles O and O', O' interior to O, then (with notation as in Theorem* 1)

$$t_{13}(t_{14}t_{34} + t_{12}t_{23}) - t_{24}(t_{14}t_{12} + t_{23}t_{34}) = 0.$$

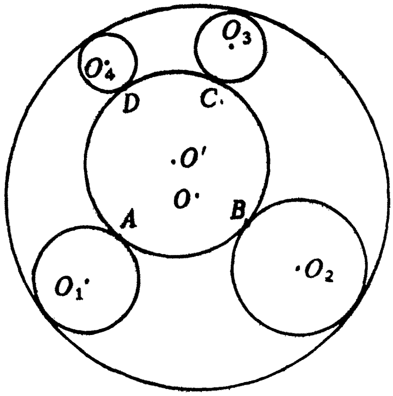


FIG. 5.

Proof. Starting from the relation

$$AC(AD \cdot CD + AB \cdot BC) - BD(AD \cdot AB + BC \cdot CD) = 0$$

and substituting from relations (2)-(7) we easily obtain

$$t_{13}[t_{14}t_{34}(R - r_2) + t_{12}t_{23}(R - r_4)] - t_{24}[t_{14}t_{12}(R - r_3) + t_{23}t_{34}(R - r_1)] = 0. \tag{14}$$

A completely analogous argument using circle O' in place of circle O leads to the relation

$$t_{13}[t_{14}t_{34}(r + r_2) + t_{12}t_{23}(r + r_4)] - t_{24}[t_{14}t_{12}(r + r_3) + t_{23}t_{34}(r + r_1)] = 0. \tag{15}$$

Adding (14) and (15) and dividing by $R + r$ yields the desired relation

$$t_{13}(t_{14}t_{34} + t_{12}t_{23}) - t_{24}(t_{14}t_{12} + t_{23}t_{34}) = 0.$$

Thus, while Casey's theorem suggests that a good analogue of four cyclic points is four circles tangent to a single circle, the failure of relation (1) to generalize with this interpretation shows that the analogy is not complete. Theorem 3 suggests that four circles satisfying the conditions of that theorem might be more properly regarded as the appropriate analogue.

The authors wish to express their gratitude to Prof. Imai and Mrs. Kimura for their kindness during the preparation of this paper.

References

1. T. Hayashi, A problem of wasan, Wasan Kenkyu Shūroku II, Kaisei Kan, Tokoyo, 1936, pp. 660–668.
2. A. Hirayama, People of wasan on record, a study as focused on arts and sciences, Fuji Junior College Press, Tokyo, 1965.
3. S. Iwata, Great Dictionary of Geometry, Maki Shoten, Tokyo 1971.
4. Y. Michiwaki, T. Hamada, M. Ōyama and K. Hatta, Some Problems on Elementary Geometry, Res. Rep. Nagaoka Tech. College, 9, 3 (1973) 127–133.
5. Y. Mikami, On Casey's theorem in Japan, Tōhoku Math. J., 15 (1919).
6. ———, The Development of Mathematics in China and Japan, Chelsea, New York, 1913.

(The above books and papers, except [4] and [6], are in Japanese.)

GENERALIZATIONS OF THE LOGARITHMIC MEAN

KENNETH B. STOLARSKY, University of Illinois, Urbana

1. Introduction. Let $G(x,y)$ and $A(x,y)$ denote the geometric and arithmetic means of the nonnegative real numbers x and y. Then

$$G(x,y) = (xy)^{\frac{1}{2}} \leqq \frac{x+y}{2} = A(x,y). \tag{1}$$

Moreover, it is an immediate consequence of (1) that if we define

$$A(x,y;1/2) = \left(\frac{x^{\frac{1}{2}} + y^{\frac{1}{2}}}{2}\right)^2 \tag{2}$$

then

$$G(x,y) \leqq A(x,y;1/2) \leqq A(x,y). \tag{3}$$

Thus $A(x,y;1/2)$ interpolates the inequality of the arithmetic and geometric means. The *logarithmic mean* $L(x,y)$, which is defined by

$$L(x,y) = \frac{x-y}{\log x - \log y}, \tag{4}$$

also has this property; that is,

$$G(x,y) \leqq L(x,y) \leqq A(x,y). \tag{5}$$

For a proof of (5), and further references on the logarithmic mean, see [3]; we comment here that it plays a role in the study of the distribution of electrical charge on a conductor ([4], p. 14).

It is well known ([1], p. 17) that if we set

$$A(x,y;\lambda) = \left(\frac{x^{\lambda} + y^{\lambda}}{2}\right)^{1/\lambda} \tag{6}$$

References

1. T. Hayashi, A problem of wasan, Wasan Kenkyu Shūroku II, Kaisei Kan, Tokoyo, 1936, pp. 660–668.
2. A. Hirayama, People of wasan on record, a study as focused on arts and sciences, Fuji Junior College Press, Tokyo, 1965.
3. S. Iwata, Great Dictionary of Geometry, Maki Shoten, Tokyo 1971.
4. Y. Michiwaki, T. Hamada, M. Ōyama and K. Hatta, Some Problems on Elementary Geometry, Res. Rep. Nagaoka Tech. College, 9, 3 (1973) 127–133.
5. Y. Mikami, On Casey's theorem in Japan, Tōhoku Math. J., 15 (1919).
6. ———, The Development of Mathematics in China and Japan, Chelsea, New York, 1913.

(The above books and papers, except [4] and [6], are in Japanese.)

GENERALIZATIONS OF THE LOGARITHMIC MEAN

KENNETH B. STOLARSKY, University of Illinois, Urbana

1. Introduction. Let $G(x,y)$ and $A(x,y)$ denote the geometric and arithmetic means of the nonnegative real numbers x and y. Then

$$G(x,y) = (xy)^{\frac{1}{2}} \leqq \frac{x+y}{2} = A(x,y). \tag{1}$$

Moreover, it is an immediate consequence of (1) that if we define

$$A(x,y;1/2) = \left(\frac{x^{\frac{1}{2}}+y^{\frac{1}{2}}}{2}\right)^2 \tag{2}$$

then

$$G(x,y) \leqq A(x,y;1/2) \leqq A(x,y). \tag{3}$$

Thus $A(x,y;1/2)$ interpolates the inequality of the arithmetic and geometric means. The *logarithmic mean* $L(x,y)$, which is defined by

$$L(x,y) = \frac{x-y}{\log x - \log y}, \tag{4}$$

also has this property; that is,

$$G(x,y) \leqq L(x,y) \leqq A(x,y). \tag{5}$$

For a proof of (5), and further references on the logarithmic mean, see [3]; we comment here that it plays a role in the study of the distribution of electrical charge on a conductor ([4], p. 14).

It is well known ([1], p. 17) that if we set

$$A(x,y;\lambda) = \left(\frac{x^{\lambda}+y^{\lambda}}{2}\right)^{1/\lambda} \tag{6}$$

for nonzero real numbers λ, and define $A(x,y;0) = G(x,y)$, then the means $A(x,y;\lambda)$ provide us with a continuous monotonic interpolation of the inequality of the arithmetic and geometric means which extends (3); more precisely, the following four conditions hold:

(i) $\min(x,y) \leqq A(x,y;\lambda) \leqq \max(x,y)$,

(ii) $A(x,y;\lambda)$ is continuous in λ,

(iii) $A(x,y;\lambda) \leqq A(x,y;\mu)$ if $\lambda \leqq \mu$,

(iv) $A(x,y;0) = G(x,y)$ and $A(x,y;1) = A(x,y)$.

The primary purpose of this paper is to do the same for (5). In the course of doing this, we shall provide yet another proof of (5).

2. A generalized logarithmic mean. To understand why $L(x,y)$ is a mean (i.e., satisfies condition (i)) simply recall that for differentiable functions f the mean value theorem asserts that for $x \neq y$ we have

$$\frac{f(x)-f(y)}{x-y} = f'(u)$$

where u is strictly between x and y. Let $f(x) = \log x$; then $u = L(x,y)$. We can now create an infinite number of "new" means simply by varying the function f. Let $f(x) = x^\alpha$ where α is a real number distinct from 0 and 1. Then

$$u = u(x,y;\alpha) = \left[\frac{x^\alpha - y^\alpha}{\alpha(x-y)}\right]^{1/(\alpha-1)}. \tag{7}$$

It is easy to verify that $u(x,y;-1) = G(x,y)$, that $\lim_{\alpha\to 0} u(x,y;\alpha) = L(x,y)$, and that $u(x,y;2) = A(x,y)$; thus our goal will be achieved, with $u = u(x,y;\alpha)$ serving as our generalized logarithmic mean, upon showing that u is monotonic in α.

At this point, however, one cannot help but wonder what happens to u as $\alpha \to 1$. Elementary calculus shows that the limit exists and equals

$$U = U(x,y) = e^{-1}(y^y x^{-x})^{1/(y-x)}. \tag{8}$$

Thus we shall also obtain a possibly new elementary inequality, namely

$$G(x,y) \leqq L(x,y) \leqq U(x,y) \leqq A(x,y). \tag{9}$$

3. Proof of monotonicity. It will be convenient to establish a monotonicity result somewhat more general than that asserted in Section 2.

DEFINITION. *For real α and β with $\alpha \neq 0$ and $\alpha \neq \beta$, and for positive x and y with $x \neq y$, let*

$$u(\alpha,\beta) = u(x,y;\alpha,\beta) = \left[\frac{\beta(x^\alpha - y^\alpha)}{\alpha(x^\beta - y^\beta)}\right]^{1/(\alpha-\beta)} \tag{10}$$

It is easily seen that $u(\alpha,\beta)$ can be extended by continuity to a function defined for all real α and β and all nonnegative x and y. For this function we have

$$u(\alpha,\beta) = u(\beta,\alpha) \tag{11}$$

$$u(x^s, y^s; \alpha, \beta)^{1/s} = u(x, y; s\alpha, s\beta), \text{ and} \tag{12}$$

$$u(\alpha,\beta)^{\alpha-\beta} = u(\alpha,\gamma)^{\alpha-\gamma}u(\gamma,\beta)^{\gamma-\beta}. \tag{13}$$

THEOREM. *For $x \neq y$, the function $u(\alpha,\beta)$ is strictly increasing in both α and β.*

Proof. By elementary calculus,

$$\begin{aligned} u(t,t) = u(x,y;t,t) &= u(x^t,y^t;1,1)^{1/t} \\ &= U(x^t,y^t)^{1/t} \\ &= \exp\left\{\frac{d}{dt}\log\left|\frac{x^t-y^t}{t}\right|\right\} \end{aligned} \tag{14}$$

and hence

$$\log u(\alpha,\beta) = \frac{1}{\beta-\alpha}\int_\alpha^\beta \log u(t,t)dt. \tag{15}$$

Thus (as is easily seen) it suffices to show that the integrand of (15) is strictly increasing. To do this it will be enough to show for $x \neq 1$ that

$$g(t) = \log\left|\frac{x^t-1}{t}\right| \tag{16}$$

is convex; i.e., that its second derivative is nonnegative. But

$$g''(t) = \frac{(z-1)^2 - z(\log z)^2}{t^2(z-1)^2}, \tag{17}$$

where $z = x^t$, and we are done, as it is easy to verify that $K(z) = (z-1)^2 - z(\log z)^2 \geqq 0$ for all $z > 0$. (For example, since $K(z^{-1}) = z^{-2}K(z)$, it suffices to show this for $z \geqq 1$. Now

$$K(z) = z^{\frac{1}{2}}(z - 1 + z^{\frac{1}{2}}\log z)(z^{\frac{1}{2}} - z^{-\frac{1}{2}} - \log z).$$

The last factor is 0 when $z = 1$ and has the nonnegative derivative $\frac{1}{2}z^{-\frac{1}{2}}[1 - z^{-\frac{1}{2}}]^2$). This completes the proof.

The equation (15) was originally discovered by replacing γ in (13) by $(\alpha+\beta)/2$, taking logarithms, iterating the resulting equation, and then passing to the limit as the number of iterations becomes infinite.

4. The main result. Our primary goal is now attained. Since

$$u(\alpha) \equiv u(x,y;\alpha) = u(x,y;\alpha,1) \tag{18}$$

we know that $u(\alpha)$ cannot decrease as α increases. Thus conditions (i)–(iv) are also satisfied for $u(3\alpha-1)$, and we have another thoroughly natural system of means interpolating the arithmetic and geometric means. Moreover, all are expressible through (15) in terms of the presumably novel mean $U(x, y)$. We also mention that as α goes to $-\infty(+\infty)$ the mean $u(\alpha)$ tends to the minimum (maximum) of x and y, and that the conditions for equality are the same as for the λth power means $A(x, y; \lambda)$.

5. More variables. The λth power means can be generalized to average any n numbers; define

$$(19) \qquad A(x_1, \cdots, x_n; \lambda) = \left(\frac{1}{n} \sum_{i=1}^{n} x_i^{\lambda}\right)^{1/\lambda}.$$

Now just as $u(x, y; \alpha)$ arose from considering a difference quotient which approximated a derivative (see Section 2), we can define a mean $u(x, y, z; \alpha)$ by using a difference quotient which approximates a second derivative:

$$(20) \qquad \begin{aligned} u(\alpha) &\equiv u(x, y, z; \alpha) \\ &= \left\{\frac{2}{\alpha(\alpha-1)}\left[\frac{z^{\alpha}(y-x)+y^{\alpha}(x-z)+x^{\alpha}(z-y)}{(z-y)(z-x)(y-x)}\right]\right\}^{1/(\alpha-2)} \end{aligned}$$

In this case the arithmetic mean arises when $\alpha = 3$ and the geometric mean when $\alpha = -1$. Moreover

$$(21) \qquad U_0 \equiv \lim_{\alpha \to 0} u(\alpha) = \left\{\frac{(z-y)(z-x)(y-x)}{2\left[x \log \frac{z}{y} + y \log \frac{x}{z} + z \log \frac{y}{x}\right]}\right\}^{\frac{1}{2}},$$

$$(22) \qquad U_1 \equiv \lim_{\alpha \to 1} u(\alpha) = \frac{(z-y)(z-x)(y-x)}{2\left[zy \log \frac{z}{y} + xz \log \frac{x}{z} + xy \log \frac{y}{x}\right]},$$

and $U_2 = \lim_{\alpha \to 2} u(\alpha)$ satisfies

$$(23) \qquad \log U_2 = -\frac{3}{2} + \frac{z^2 \log z}{(z-x)(z-y)} + \frac{y^2 \log y}{(y-x)(y-z)} + \frac{x^2 \log x}{(x-y)(x-z)}.$$

Here U_2 is analogous to our old $U(x, y)$ while U_0 and U_1 provide us with *two* generalizations of the logarithmic mean. If we let

$$(24) \qquad I = \int_0^{\infty} \frac{dt}{(t+x)(t+y)(t+z)}$$

then $U_0(x, y, z) = (2I)^{-\frac{1}{2}}$ and the fact that U_0 separates the arithmetic and geometric means follows easily from

$$(t + (xyz)^{1/3})^3 \leqq (t+x)(t+y)(t+z) \leqq \left(t + \frac{x+y+z}{3}\right)^3$$

(this sort of trick occurs in Szegö [**5**] and Carlson ([**3**], p. 615 and [**2**], p. 701)). We also note that

$$(25) \qquad xyz \cdot U_1(x, y, z) = U_0^2(xy, xz, yz).$$

We conjecture that

$$xyz \cdot U_0(x,y,z) \leqq U_0^2(xy,xz,yz); \tag{26}$$

if U_0 is replaced by the arithmetic mean in (26), a known inequality is obtained (see, e.g., [1], p. 11, inequality (6)).

At this point we attempt to generalize $u(\alpha)$ as far as possible. Let $x_1, \cdots, x_{n+1}$ be positive numbers and set

$$a_i(t) = x_i^t \prod_{1 \leqq j < k \leqq n+1}' (x_j - x_k) \tag{27}$$

where the prime mark indicates that every factor involving x_i is deleted. Let $(\alpha)_n = \alpha(\alpha-1)\cdots(\alpha-n+1)$. Define

$$\begin{aligned} u(\alpha,\beta) &= u(x_1,\cdots,x_{n+1};\alpha,\beta) \\ &= \left\{\left[(\beta)_n \sum_{i=1}^{n+1} (-1)^{i+1} a_i(\alpha)\right] \Big/ \left[(\alpha)_n \sum_{i=1}^{n+1} (-1)^{i+1} a_i(\beta)\right]\right\}^{1/(\alpha-\beta)} \end{aligned} \tag{28}$$

It is easy to see that this reduces to the earlier definitions when $n = 1$ or 2. Our main problem is to show that $u(\alpha,\beta)$ cannot decrease if either α or β is increased. Once again we have (15) where now

$$\log u(t,t) = \frac{d}{dt} G_n(t) \text{ and} \tag{29}$$

$$G_n(t) = \log\left|\frac{\sum_{i=1}^{n+1} (-1)^{i+1} a_i(t)}{(t)_n}\right|. \tag{30}$$

Conjecture. The function $G_n(t)$ is convex.

I have not been able to resolve this conjecture; the second derivative of $G_n(t)$ is unwieldy. I can, however, manipulate

$$G_n''(t) \geqq 0 \tag{31}$$

into a sort of "standard form" with the aid of the identity

$$\left(\sum_{i=1}^{n} A_i B_i^2\right)\left(\sum_{j=1}^{n} A_j\right) - \left(\sum_{i=1}^{n} A_i B_i\right)^2 = \sum_{i<j} A_i A_j (B_i - B_j)^2. \tag{32}$$

The result is that (31) is equivalent to

$$\begin{aligned} &\left(\sum_{i=1}^{n+1} (-1)^{i+1} a_i(t)\right)^2 \left(\sum_{j=0}^{n-1} \frac{1}{(t-j)^2}\right) \\ &\quad + \sum_{i<j} (-1)^{i+j} a_i(t) a_j(t) (\log x_i - \log x_j)^2 \geqq 0. \end{aligned} \tag{33}$$

The inequality (33) has a "vague reasonableness" to it. The double sum can be positive or negative, but the other terms are clearly nonnegative. The first sum on the left vanishes when $t = 0, 1, \cdots, n-1$ but the second sum compensates by having double poles at these points. For $n = 1$, the inequality (33) quickly reduces to a homogeneous form of the known inequality $(u-1)^2 \geqq u(\log u)^2$. For $n = 2$, if we divide (33) by $(x_1 - x_3)^2$ and take the limit as $x_1 \to x_3$ we obtain the (unproved) statement

$$(34) \quad \begin{aligned} &[(t-1)u^t - tu^{t-1}]\log^2 u + 2(u^t - u^{t-1})\log u - u^{2(t-1)}(u-1)^2 \\ &\qquad \leqq \left(\frac{1}{t^2} + \frac{1}{(t-1)^2}\right)(tu^{t-1} - (t-1)u^t - 1)^2. \end{aligned}$$

Note that (34) is unchanged if u and t are replaced by u^{-1} and $1-t$. If we denote the left hand side of (33) by $\sigma = \sigma(x_1, \cdots, x_{n+1}; t)$ then

$$\sigma(x_1, \cdots, x_{n+1}; t) = (x_1 \cdots x_{n+1})^{2(n-1)} \sigma(x_1^{-1}, \cdots, x_{n+1}^{-1}; n-1-t).$$

It is easy to see that (33) is true for $|t|$ large; we also mention that

$$\lim_{t \to 0} \sigma(x_1, x_2, x_3; t) = 0$$

and hence also $\lim_{t \to 1} \sigma = 0$ when $n = 2$.

6. Comment. We note that the mean $u(x, y; \alpha)$ bears some resemblance to the functions denoted by $G_t(x, y)$ and $A_t(x, y)$ in Carlson's paper ([3], p. 616); note especially his remark there that "$1/G_t$ is log convex in t".

References

1. E. F. Beckenbach and R. Bellman, Inequalities, Springer-Verlag, New York, 1962.
2. B. C. Carlson, Inequalities for a symmetric elliptic integral, Proc. Amer. Math. Soc., 25 (1970) 698–703.
3. ———, The logarithmic mean, Amer. Math. Monthly, 79 (1972) 615–618.
4. G. Polya and G. Szegö, Isoperimetric Inequalities in Mathematical Physics, Princeton University Press, Princeton, 1951.
5. G. Szegö, Solution of Problem 542, Arch. Math. und Physik, (3) 28 (1920) 81–82.

AN EXTENSION OF TRIGG'S TABLE

SIDNEY KRAVITZ, Dover, N.J. and DAVID E. PENNEY, University of Georgia

In a recent issue of this MAGAZINE, Charles W. Trigg asks in [1] some interesting questions about the prime factorization of

$$Q(p_k) = (p_1 p_2 p_3 \cdots p_k) + 1$$

where p_i denotes the ith prime. He tabulated the prime factorization of $Q(p)$ for $2 \leqq p \leqq 19$; we include and extend his work as summarized in Table 1, page 93. We also show data on the closely related

$$R(p_k) = (p_1 p_2 p_3 \cdots p_k) - 1.$$

double poles at these points. For $n = 1$, the inequality (33) quickly reduces to a homogeneous form of the known inequality $(u-1)^2 \geqq u(\log u)^2$. For $n = 2$, if we divide (33) by $(x_1 - x_3)^2$ and take the limit as $x_1 \to x_3$ we obtain the (unproved) statement

$$(34) \qquad \begin{aligned}[(t-1)u^t - tu^{t-1}]\log^2 u + 2(u^t - u^{t-1})\log u - u^{2(t-1)}(u-1)^2 \\ \leqq \left(\frac{1}{t^2} + \frac{1}{(t-1)^2}\right)(tu^{t-1} - (t-1)u^t - 1)^2.\end{aligned}$$

Note that (34) is unchanged if u and t are replaced by u^{-1} and $1 - t$. If we denote the left hand side of (33) by $\sigma = \sigma(x_1, \cdots, x_{n+1}; t)$ then

$$\sigma(x_1, \cdots, x_{n+1}; t) = (x_1 \cdots x_{n+1})^{2(n-1)}\sigma(x_1^{-1}, \cdots, x_{n+1}^{-1}; n - 1 - t).$$

It is easy to see that (33) is true for $|t|$ large; we also mention that

$$\lim_{t \to 0} \sigma(x_1, x_2, x_3; t) = 0$$

and hence also $\lim_{t \to 1} \sigma = 0$ when $n = 2$.

6. Comment. We note that the mean $u(x, y; \alpha)$ bears some resemblance to the functions denoted by $G_t(x, y)$ and $A_t(x, y)$ in Carlson's paper ([3], p. 616); note especially his remark there that "$1/G_t$ is log convex in t".

References

1. E. F. Beckenbach and R. Bellman, Inequalities, Springer-Verlag, New York, 1962.
2. B. C. Carlson, Inequalities for a symmetric elliptic integral, Proc. Amer. Math. Soc., 25 (1970) 698–703.
3. ———, The logarithmic mean, Amer. Math. Monthly, 79 (1972) 615–618.
4. G. Polya and G. Szegö, Isoperimetric Inequalities in Mathematical Physics, Princeton University Press, Princeton, 1951.
5. G. Szegö, Solution of Problem 542, Arch. Math. und Physik, (3) 28 (1920) 81–82.

AN EXTENSION OF TRIGG'S TABLE

SIDNEY KRAVITZ, Dover, N.J. and DAVID E. PENNEY, University of Georgia

In a recent issue of this MAGAZINE, Charles W. Trigg asks in [1] some interesting questions about the prime factorization of

$$Q(p_k) = (p_1 p_2 p_3 \cdots p_k) + 1$$

where p_i denotes the ith prime. He tabulated the prime factorization of $Q(p)$ for $2 \leqq p \leqq 19$; we include and extend his work as summarized in Table 1, page 93. We also show data on the closely related

$$R(p_k) = (p_1 p_2 p_3 \cdots p_k) - 1.$$

TABLE 1

p	$Q(p)$	$R(p)$
2	3	1
3	7	5
5	31	29
7	211	11·19
11	2311	2309
13	59·509	30029
17	19·97·277	61·8369
19	347·27953	53·197·929
23	317·703763	37·131·46027
29	331·571·34231	79·81894851
31	200560490131	228737·876817
37	181·60611·676421	229·541·1549·38669
41	61·450451·11072701	304250263527209
43	167·78339888213593	141269·92608862041
47	953·46727·13808181181	191·53835557·59799107
53	73·139·173·18564761860301	87337·326257·1143707681
59	277·3467·105229·19026377261	C_2
61	223·525956867082542470777	1193·C_2
67	C_3	163·2682037·1797535293624551 9
71	1063·303049·598841·2892214489673	C_3
73	2521·P_3	313·1301267750774729206090138 13
79	22093·C_3	163·2843·C_3
83	265739·P_2	139·26417·P_2
89	131·1039·2719·64225891884294373371806141	2376874189634555077065053760 1358309
97	2336993·C_4	66683·P_3

In Table 1, all entries for $Q(p)$ and $R(p)$ are primes or 1, except that C_n denotes a composite number with no more than n prime factors and P_n denotes a number, possibly prime, with no more than n prime factors. None of the C_n and P_n in Table 1 has a prime factor less than 10^7, and all were checked for divisors sufficiently large to establish the validity of the subscript. The P_n satisfy the congruence

$$2^{m-1} \equiv 1 \pmod{m}$$

and thus are quite likely prime; indeed, we were able to establish some of the larger factors prime by an application of a version of a theorem of Lehmer [**2**]:

THEOREM. *Let b and n be integers exceeding* 1. *Suppose that* $b^{n-1} \equiv 1 \pmod{n}$, *and let p be a prime factor of* $n-1$. *Let* $a \equiv b^{(n-1)/p} \pmod{n}$. *If* $(n, a-1) = 1$, *then every prime factor q of n satisfies* $q \equiv 1 \pmod{p}$.

We owe special thanks to Dr. Carl Pomerance of the University of Georgia, who designed an eminently programmable version of this test.

We also obtained data on $Q(p)$ and $R(p)$ for larger values of the prime p, and we obtained coincident results although at the time we were working independently, with different programs, on different computers, each of us unaware of the other's work. With the aid of Table 1, special-purpose programs, and some recent results in the literature, we can answer most of Trigg's questions.

1. Are any $Q(p)$ prime for $p > 19$?

This question was answered by Kraitchik [3], and his results extended by Borning [4], who found that in the range $23 \leqq p \leqq 307$, only $Q(31)$ is prime. Borning also found that for $p \leqq 307$, $R(p)$ is prime only for $p = 3, 5, 11, 13, 41$, and 89. We have confirmed these results for $p \leqq 97$, and Table 1 also gives complete or partial factorizations not given by Kraitchik or Borning.

2. The prime $p_7 = 17$ and the least prime factor $p_8 = 19$ of $Q(17)$ are twin primes. Does this case of twin primes, or even of consecutive primes, occur again?

Yes; $Q(1459)$ is divisible by $p_{233} = 1471$, but the latter and $p_{232} = 1459$ are not twin primes. In the range $19 < p \leqq p_{6000} = 59359$, there is only one other such example: $Q(2999)$ is divisible by $p_{431} = 3001$, and the latter and $p_{430} = 2999$ do form a twin prime pair.

The same question for $R(p)$ leads to the obvious examples for $p = 3$ and $p = 7$; there are no other examples for which p_{k+1} is a divisor of $R(p_k)$ in the abovementioned range. There are a few cases in which the second or third prime after p divides $Q(p)$ or $R(p)$—specifically, $7 \mid Q(3)$, $37 \mid R(23)$, $271 \mid Q(263)$, $307 \mid Q(283)$, and $673 \mid Q(659)$. There are no additional examples in the range $p \leqq 59359$.

3. Are there more cases in which the least prime factor of $Q(p)$ does not exceed $2p$?

This holds for $p = 2$ and for $p = 17$, as observed by Trigg. We found it to hold for exactly 32 values of p in the range $2 \leqq p \leqq 1987$, and the same holds true for $R(p)$ for 24 such values. These are shown in Table 2, together with the divisor or divisors less than $2p$.

4. What is the smallest value of p for which $Q(p)$ has four prime factors? Five prime factors?

$Q(53)$ is the least value of $Q(p)$ with four prime factors, and has exactly four. We found none with five prime factors, and $Q(97)$ is the least candidate for this property. $R(37)$ has exactly four prime factors, and is the least value of $R(p)$ with at least four; $R(79)$ might have as many as five.

In the course of these investigations some additional facts were noted. We mention three here:

TABLE 2

p	Prime divisors of $Q(p)$ not exceeding $2p$	p	Prime divisors of $R(p)$ not exceeding $2p$
2	3	3	5
17	19	7	11
41	61	23	37
53	73	83	139
89	131	167	331
107	149	239	349
239	313	241	389
263	271	397	599
283	307	421	761
443	463	463	631 and 647
499	827	499	569
587	1033	523	563
659	673	577	1093
677	809 and 877	641	881
739	1051	797	953
769	997 and 1297	877	911
811	1279	907	983
839	1109	919	1181
907	1259	941	1433
937	1031	1069	1327
1061	2029	1103	1283
1097	1381	1289	1811
1181	1667	1871	3467
1237	1663	1877	2531
1259	1867		
1423	2609		
1459	1471		
1481	1619		
1657	3203		
1663	2383		
1669	3041		
1987	3581		

First, some primes—for example, 13, 17, 23, and 41—divide none of the $Q(p)$ and none of the $R(p)$.

Second, several primes may divide two values of $Q(p)$, two values of $R(p)$, or one of each. All such between 2 and $p_{1001} = 7927$ are shown in Table 3, together with the $Q(p)$ and $R(p)$ they divide for $p \leqq 7919$.

TABLE 3

p	What p divides	p	What p divides
19	$Q(17)$ and $R(7)$	1051	$Q(211)$ and $Q(739)$
61	$Q(41)$ and $R(17)$	1069	$Q(523)$ and $R(359)$
131	$Q(89)$ and $R(23)$	1283	$Q(509)$ and $R(1103)$
139	$Q(53)$ and $R(83)$	1291	$Q(439)$ and $R(163)$
163	$R(67)$ and $R(79)$	1381	$Q(157)$ and $Q(1097)$
277	$Q(17)$ and $Q(59)$	1657	$Q(137)$ and $Q(557)$
313	$Q(239)$ and $R(73)$	1867	$Q(157)$ and $Q(1259)$
331	$Q(29)$ and $R(167)$	2609	$Q(1423)$ and $R(479)$
673	$Q(659)$, $R(149)$, and $R(193)$	3041	$Q(1277)$ and $Q(1669)$
881	$Q(137)$ and $R(641)$	3373	$Q(521)$ and $Q(1103)$
953	$Q(47)$ and $R(797)$	3467	$Q(59)$ and $R(1871)$
983	$Q(463)$ and $R(907)$	4871	$Q(613)$ and $R(139)$

Finally, we checked $Q(p)$ and $R(p)$ for prime factors less than 10^7 for $2 \leqq p \leqq 97$, for prime factors less than 10^5 for $101 \leqq p \leqq 541 = p_{100}$, and for prime factors less than 7930 for $547 \leqq p \leqq 1987$. As a result we know that $Q(p)$ is prime for six values of p, composite for 106 values of p, and unknown to us for the remaining 188 values of p. Similarly, $R(p)$ is a unit for $p = 2$, prime for six values of p, composite for 96 values of p, and unknown to us for the remaining 197 values of p. The largest number we actually computed was

$$Q(59359) = 62970292 \cdots 375361614691,$$

a number of 25706 digits.

Questions inevitably remain. What is the least value of $Q(p)$ having exactly (or at least) five prime factors? Or, six, or seven? Are any more of the $Q(p)$ prime? What are the answers to these questions for the $R(p)$? Note that $R(p)$ and $Q(p)$ form a twin prime pair for $p = 3, 5, 11$, and for no other prime $p \leqq 307$. Is there another such twin prime pair? Are there infinitely many primes dividing none of the $Q(p)$ and none of the $R(p)$? It is easy to show that none of the $Q(p)$ and none of the $R(p)$ can be perfect squares other than $R(2)$. We know all are square-free for $p \leqq 61$. Does this hold for all p?

References

1. C. W. Trigg, The infinitude of primes, this MAGAZINE, 47 (1974) 162.

2. D. H. Lehmer, Computer technology applied to the theory of numbers, in Studies in Number Theory (W. J. LeVeque, editor), Prentice-Hall, Englewood Cliffs, New Jersey, 1969, p. 128.

3. Maurice Kraitchik, Introduction à la Théorie des Nombres, Gauthier-Villars, Paris, 1952, pp. 2, 8.

4. Alan Borning, Some results for $k! \pm 1$ and $2 \cdot 3 \cdot 5 \cdots p \pm 1$, Math. Comp., 26 (1972) 567–570.

BIPERIODIC SQUARES

FRITZ HERZOG, Michigan State University

In this paper we use an arbitrary scale of enumeration to the base B, where $B \geqq 2$. We use the notation

$$\sum_{j=0}^{h-1} b_j B^j = (b_{h-1}b_{h-2} \cdots b_1 b_0)_B. \tag{1}$$

The digits b_j are integers in the range $0 \leqq b_j < B$. Whenever no base B is indicated it is understood that we deal with the ordinary decimal scale, $B = 10$.

The number in (1) will be called *biperiodic* (having two periods) if its digits form two equal halves, that is, if it is of the form

$$(a_{m-1}a_{m-2} \cdots a_1 a_0 a_{m-1} a_{m-2} \cdots a_1 a_0)_B. \tag{2}$$

We require here as usual that the first digit a_{m-1} should not be zero. It will, however, be convenient to consider at first also numbers of the form (2) with $a_{m-1} = 0$; these will be called *improperly* biperiodic. The term "biperiodic" (without preceding adverb) will thus be reserved for the case where $a_{m-1} > 0$. For example, 11 and 614614 are biperiodic, while 202 and 510051 are improperly biperiodic.

The problem discussed in this paper is the following: In a given scale of enumeration, are there biperiodic squares? Do we, for instance, encounter eventually biperiodic numbers in the decimal scale among the numbers $1, 4, 9, 16, 25, 36, \cdots$? An inspection of Barlow's Table (see Reference [1]) which gives the squares of all integers up through 12500 reveals the existence of no such square within the limits of the Table. On the other hand, in scales other than the decimal one, we have many examples of biperiodic squares at a fairly low level, such as $(100100)_2 = 6^2, (11)_3 = 2^2, (211211)_5 = 84^2, (1001)_2 = 3^2, (202102021)_3 = 122^2$. Note that the last two examples are improperly biperiodic.

In analyzing our problem, let us assume first that the number in (2) which we shall denote by Q is a square. We do not specify at this point that $a_{m-1} > 0$. If we set $(a_{m-1}a_{m-2} \cdots a_1 a_0)_B = b$ we have

$$Q = b(B^m + 1), \tag{3}$$

where

$$0 < b < B^m. \tag{4}$$

Now, if $B^m + 1$ is square-free, which means that $B^m + 1$ is not divisible by any square greater than 1, then in order for Q in (3) to be a square it would be necessary that b be equal to $(B^m + 1)$ times a square. In any case $b \geqq B^m + 1$ which is incompatible with (4).

Before proceeding, we introduce the following notation: Let

$$B^m + 1 = s \cdot t^2, \; s \text{ square-free}. \tag{5}$$

The requirement that s be square-free is equivalent to saying that t^2 should be the largest square dividing $B^m + 1$. Thus s and t are uniquely determined by (5), and $B^m + 1$ is square-free if and only if $t = 1$.

We assume now that $B^m + 1$ is not square-free, that is, $t > 1$. By (3) and (5), $Q = bst^2$ and thus Q will be a square if and only if $b = s \cdot k^2$ for some positive integer k. By (4), we now have $s \cdot k^2 = b < B^m < B^m + 1 = s \cdot t^2$, hence $k < t$, and k must be one of the values $1, 2, \cdots, t-1$. Conversely, any of these values leads to a value of b that satisfies (4), because if $k \leqq t-1$ then $b = s \cdot k^2 \leqq s(t-1)^2 = s \cdot t^2 - s(2t-1) = B^m + 1 - s(2t-1) \leqq B^m + 1 - 3 < B^m$. We have thus arrived at the following theorem which answers our problem, provided no distinction is made between biperiodic squares and improperly biperiodic ones.

THEOREM 1. *Given $B \geqq 2$ and $m \geqq 1$. There exist biperiodic squares of the form* (2), *improperly or not, if and only if $B^m + 1$ is not square-free. If this condition is satisfied then one obtains all biperiodic squares with the given values of B and m by setting $b = s \cdot k^2$, $k = 1, 2, \cdots, t-1$, in* (3), *where s and t are defined by* (5).

Examples: (a) $B = 10, m = 3.$ $10^3 + 1 = 7 \cdot 11 \cdot 13$ is square-free. There are no biperiodic squares with $B = 10$, $m = 3$.

(b) $B = 4, m = 5.$ $4^5 + 1 = 41 \cdot 5^2$; $s = 41, t = 5.$ $b = 41k^2, k = 1, 2, 3, 4.$ The resulting biperiodic squares are: $(22100221)_4 = 205^2$, $(221002210)_4 = 410^2$, $(1130111301)_4 = 615^2$, $(2210022100)_4 = 820^2$. Note that the first two of these are improperly biperiodic.

Judging from Example (b), it seems, indeed, desirable to eliminate from our considerations those squares which are only improperly biperiodic. We assume accordingly that $B^m + 1$ is not a square-free number. Among the squares obtained in Theorem 1, those will be improperly biperiodic for which $a_{m-1} = 0$ in (2) which means, by virtue of the definition of b, those for which $b = s \cdot k^2 < B^{m-1}$; and those for which $a_{m-1} > 0$ or $b = s \cdot k^2 \geqq B^{m-1}$ will be biperiodic. For given values of B and m for which $B^m + 1$ is not square-free, are there always biperiodic squares?

Since $b = s(t-1)^2$ is, by Theorem 1, the largest possible value of b the answer to the above question is yes if and only if

$$s(t-1)^2 \geqq B^{m-1}. \tag{6}$$

We will show that (6) is satisfied when either $B \geqq 4$ or $t \geqq 4$. We have

$$s(t-1)^2 = \left(\frac{t-1}{t}\right)^2 \cdot st^2 = \left(\frac{t-1}{t}\right)^2 (B^m + 1) > \left(\frac{t-1}{t}\right)^2 B^m$$

and

$$\left(\frac{t-1}{t}\right)^2 B^m \geqq \begin{cases} \frac{1}{4}B^m \geqq B^{m-1}, & \text{if } B \geqq 4; \\ \frac{9}{16}B^m > \frac{1}{2}B^m \geqq B^{m-1}, & \text{if } t \geqq 4. \end{cases}$$

The exceptional cases are $B = 2$, $t = 3$, and $B = 3$, $t = 2$. [Note that, by (5), B and t must be relatively prime.] In the first case, we have $2^m + 1 = s \cdot 3^2$, the inequality (6) becomes $4(2^m + 1)/9 \geqq 2^{m-1}$ or $2^{m-1} \leqq 4$, which is satisfied when and only when $m \leqq 3$. In the second case we have $3^m + 1 = s \cdot 2^2$; (6) becomes $(3^m + 1)/4 \geqq 3^{m-1}$ or $3^{m-1} \leqq 1$ which is satisfied when and only when $m = 1$. We thus obtain the following theorem in answer to our question.

THEOREM 2. *Given $B \geqq 2$ and $m \geqq 1$, and assume that the condition of Theorem 1 is satisfied, that is, that $B^m + 1$ is not square-free. Among the squares of the form (2) obtained in Theorem 1 those and only those are biperiodic for which $b = s \cdot k^2 \geqq B^{m-1}$. There exist such biperiodic squares in all cases where $B^m + 1$ is not square-free, except when $B = 2$, $m > 3$, $t = 3$, and $B = 3$, $m > 1$, $t = 2$.*

The table below is based entirely on Theorems 1 and 2. It lists, for all values of $B \leqq 10$, those values of m for which $B^m + 1$ is not square-free and less than 100,000. It also gives the values of s and t, according to (5), and the values of k from 1 through $t - 1$. Among the latter, those enclosed in parentheses give rise to improperly biperiodic squares, those not in parentheses yield biperiodic squares. All entries are written in the decimal scale.

TABLE

B	m	s	t	k
2	3	1	3	(1), 2
2	9	57	3	(1, 2)
2	10	41	5	(1, 2, 3), 4
2	15	3641	3	(1, 2)
3	1	1	2	1
3	3	7	2	(1)
3	5	61	2	(1)
3	7	547	2	(1)
3	9	4921	2	(1)
3	10	2362	5	(1, 2), 3, 4
4	5	41	5	(1, 2), 3, 4
5	3	14	3	(1), 2
7	1	2	2	1
7	2	2	5	(1), 2, 3, 4
7	3	86	2	1
7	5	4202	2	1
8	1	1	3	1, 2
8	3	57	3	(1), 2
8	5	3641	3	(1), 2
9	5	2362	5	(1), 2, 3, 4

We finally ask if, for a given value of $B \geqq 2$, there always are biperiodic squares. More precisely, we shall show that for each $B \geqq 2$ there are infinitely many values of m which make $B^m + 1$ not square-free and, moreover, that in the cases $B = 2$ and 3 there are infinitely many values of m which make $B^m + 1$ divisible by a square greater than or equal to 4^2. The statement in the last sentence, according to Theorem 2, ensures the existence of infinitely many biperiodic squares in each base $B \geqq 2$. It suffices, however, in all cases to exhibit only one value of m of the desired kind. For, if u is any odd integer, $u \geqq 3$, then

$$B^{um} + 1 = (B^m + 1)[B^{(u-1)m} - B^{(u-2)m} + \cdots + B^{2m} - B^m + 1]$$

which shows that any square dividing $B^m + 1$ also divides $B^{um} + 1$ for $u = 3, 5, 7, \cdots$.

Now, for $B = 2$ and 3 we only have to point out that $2^{10} + 1$ and $3^{10} + 1$ are divisible by 5^2. If B is of the form $4j - 1, j \geqq 2$, then $B + 1 = 4j$ and is divisible by 2^2. In all other cases $B + 1 \geqq 5$ and is not divisible by 4, hence $B + 1$ has an odd divisor $d \geqq 3$. Let $B + 1 = d \cdot e$. We then have, with $C_{n,k}$ denoting the binomial coefficients,

$$\begin{aligned} B^d + 1 &= (de - 1)^d + 1 = (de)^d - C_{d,1}(de)^{d-1} \\ &\quad + C_{d,2}(de)^{d-2} - \cdots - C_{d,d-2}(de)^2 + C_{d,d-1}(de) - 1 + 1. \end{aligned} \tag{7}$$

After cancelling the last two terms on the right-hand side of (7), the remaining terms are all divisible by d^2, because $C_{d,d-1} = d$. Thus (7) shows that $B^d + 1$ is divisible by d^2, and we have arrived at the following theorem:

THEOREM 3. *In any scale of enumeration $B \geqq 2$ there are infinitely many biperiodic squares.*

According to Theorem 3, there are then in particular in the ordinary decimal scale infinitely many biperiodic squares, and it is appropriate to ask for the smallest such square. For this purpose we have, by Theorems 1 and 2, first to find the smallest value of m for which $10^m + 1$ fails to be square-free, and then to find the smallest value of k for which $s \cdot k^2 \geqq 10^{m-1}$. Now Rollett (see Reference [2]) has given the complete factorizations of all numbers $10^m + 1$ for $1 \leqq m \leqq 18$. Only one of them is not square-free, namely $10^{11} + 1 = 11^2 \cdot 23 \cdot 4093 \cdot 8779$ for which $s = 826,446,281$ and $t = 11$. [The fact that $10^{11} + 1$ must be divisible by 11^2 follows incidentally from the considerations made above in connection with (7).] The smallest value of k for which $826,446,281 \cdot k^2 \geqq 10^{10}$ is $k = 4$. Thus the smallest biperiodic square in the decimal scale is

$$13223140496, 13223140496 = (36, 363, 636, 364)^2.$$

References

1. Peter Barlow, Barlow's Tables of Squares, Cubes, Square Roots, Cube Roots and Reciprocals of all Integers up to 12500, 4th ed., Chemical Publishing Company, Brooklyn, New York, 1954.

2. A. P. Rollett, The factors of 11... 11, etc., Math. Gaz., 21 (1937) 410–411.

REMARKS ON LIMITS OF FUNCTIONS

R. B. DARST and E. R. DEAL, Colorado State University

The examples $f_1(x, y) = xy(x^2 + y^2)^{-1}$ and $f_2(x, y) = x^2y(x^4 + y^2)^{-1}$ are often used in advanced calculus classes when discussing limits of real valued functions defined on a region in the plane. More generally, the function $f_k(x, y) = x^k y(x^{2k} + y^2)^{-1}$ has limit zero at $(0, 0)$ along the graph of a polynomial $y = p(x)$ of degree less than k in x but has value $1/2$ on $y = x^k$, $x \neq 0$. Furthermore, $f(x, y) = e^{(-1/x^2)} y(e^{(-2/x^2)} + y^2)^{-1}$ has limit zero along every polynomial path $y = p(x)$. However, these examples are, after substitution and relabeling, all of the form $xy(x^2 + y^2)^{-1}$ and, thus, a special case of $xy(x^a + y^b)^{-1}$, $a, b > 1$, $x, y > 0$. To investigate this latter function near $(0, 0)$, set $x = y^p$, $p > 0$, and obtain $y^{(1+p)}(y^{ap} + y^b)^{-1}$, which is near zero unless $1 + p \leqq ap$ and $1 + p \leqq b : (a-1)^{-1} \leqq p \leqq b - 1$. Thus, the limit at $(0, 0)$ is zero unless $1 \leqq (a-1)(b-1) : ab \geqq a + b$, so these two dimensional examples are very easy to analyze.

After the preceding discussion, it is interesting to investigate the situation in higher dimensions. For instance, what can one say about $xyz(x^2 + y^3 + z^4)^{-1}$ in the positive octant near the origin? We shall formulate the problem in n dimensions and show that it has a nice geometric interpretation. Then we shall consider the case $n = 3$ in detail, but merely state the solution in the general case. Thus, we set $x = \{x_i\}_{i=1}^n$, $x_n = y$, $a_n = b$, $x_i = y^{p_i}$, $x_i, p_i > 0$, $a_i > 1$, $i = 1, \cdots, n-1$, and

$$f(x) = (\Pi_{i \leqq n} x_i)(\Sigma_{i \leqq n} x_i^{a_i})^{-1} = y^{(1 + \Sigma_{i \leqq n} p_i)}(y^{a_1 p_1} + y^{a_2 p_2} + \cdots + y^b)^{-1}.$$

In order that $\lim_{x \to 0^+} f(x) \neq 0$, it is necessary and sufficient that the system $1 + \Sigma_{j<n} p_j \leqq a_i p_i$, $i < n$, $\Sigma_{j<n} p_j \leqq b - 1$ of n linear inequalities in $p_1, \cdots, p_k$, $k = n - 1$ have an admissible solution. To analyze these inequalities, notice that the admissible solutions of the first k comprise a cone (perhaps empty), and the whole system has solutions if, and only if, the vertex of that cone satisfies the last inequality. This interpretation permits us to replace k inequalities by a system of k linear equations. Now we let $n = 3$ as promised and solve the system $1 + p_1 + p_2 = a_i p_i$, $i = 1, 2$ to find that $(p_1, p_2) = [a_1 a_2 - (a_1 + a_2)]^{-1}(a_2, a_1)$ is the vertex of the cone. This vertex is admissible if $a_1 a_2 - (a_1 + a_2) > 0$, and it satisfies the last necessary inequality if $p_1 + p_2 = (a_1 + a_2)[a_1 a_2 - (a_1 + a_2)]^{-1} \leqq a_3 - 1$: $a_1 a_2 a_3 - (a_1 a_2 + a_1 a_3 + a_2 a_3) \geqq 0$. For example, if $a_1 = 2$, $a_2 = 3$, $a_3 = 4$, then $6 - 5 > 0$ but $24 - (6 + 8 + 12) < 0$, so $\lim_{(x,y,z) \to 0^+} xyz(x^2 + y^3 + z^4)^{-1} = 0$.

We leave the reader with the exercise of verifying that for $n \geqq 3$, $(p_1, \cdots, p_k)$ is admissible and $\Sigma p_j \leqq b - 1 = a_n - 1$ if, and only if,

(i) $(\Pi_{j<n} a_j - \Sigma_{i<n} \Pi_{i \neq j < n} a_j) > 0$ and

(ii) $(\Pi_{j \leqq n} a_i - \Sigma_{i \leqq n} \Pi_{i \neq j \leqq n} a_j) \geqq 0$.

A NOTE ON CONGRUENCE PROPERTIES OF CERTAIN RESTRICTED PARTITIONS

DEAN R. HICKERSON, University of California, Davis

K. Thanigasalam proved in this MAGAZINE (see [1]) the following theorem relating congruence properties of $p(n)$, the number of unrestricted partitions of a natural number n, to those of $p_k^*(n)$, the number of partitions of n into parts not divisible by k, where $k \geqq 2$ is a given natural number:

THEOREM 1. *Let* $k \geqq 2$, *and* l *be an integer satisfying* $0 \leqq l \leqq k-1$. *If* $p(km + l) \equiv 0 \pmod{k}$ *for* $m=0,1,2,\cdots$, *then* $p_k^*(km + l) \equiv 0 \pmod{k}$ $(m = 0, 1, 2, \cdots)$.

The author then states and proves the converse.

In this note, we present a more general theorem and a shorter proof than the one given in [1]; in particular, we do not use Euler's product formula.

If S is a set of positive integers and n an integer, denote by $p_S(n)$ the number of partitions of n with parts in S.

THEOREM 2. *Let* k *be a positive integer and suppose the integers in* S *not divisible by* k *are the same as the integers in* T *not divisible by* k. *If* $p_S(n) \equiv 0 \pmod{k}$ *for all* $n \equiv l \pmod{k}$, *then* $p_T(n) \equiv 0 \pmod{k}$ *for all* $n \equiv l \pmod{k}$.

Proof. The generating function for p_T is

$$\sum_n p_T(n)x^n = \prod_{\substack{r \\ r\in T}} (1-x^r)^{-1} = \prod_{\substack{r \\ r\in T \\ k\nmid r}} (1-x^r)^{-1} \prod_{\substack{r \\ r\in T \\ k\mid r}} (1-x^r)^{-1}$$

$$= \prod_{\substack{r \\ r\in S}} (1-x^r)^{-1} \prod_{\substack{r \\ r\in S \\ k\mid r}} (1-x^r) \prod_{\substack{r \\ r\in T \\ k\mid r}} (1-x^r)^{-1} = \left(\sum_n p_S(n)x^n\right)\left(\sum_n a_n x^n\right),$$

where

$$\sum_n a_n x^n = \prod_{\substack{r \\ r\in S \\ k\mid r}} (1-x^r) \prod_{\substack{n \\ r\in T \\ k\mid r}} (1-x^r)^{-1}.$$

Note that $a_n = 0$ if $k \nmid n$, so, for all n,

$$p_T(n) = \sum_u a_{ku} p_S(n-ku).$$

Hence, if $n \equiv l \pmod{k}$, then $p_T(n) \equiv \sum_u a_{ku} \cdot 0 = 0 \pmod{k}$.

The special case $S = N$, $T = \{r \in N : k \nmid r\}$ gives Theorem 1 and $S = \{r \in N : k \nmid r\}$, $T = N$ its converse.

Reference

1. K. Thanigasalam, Congruence properties of certain restricted partitions, this MAGAZINE, 47 (1974) 154–156.

PRISONER'S DILEMMA, A STOCHASTIC SOLUTION

WALTER W. HILL JR., MIT, Cambridge, Mass.

I would like to analyze the two-person game *Prisoner's Dilemma* from a slightly different perspective than that found in the traditional literature. By redefining the game, we will find an equilibrium strategy at the maximum payoff.

First assume the following payoff matrix:

	Deny	Confess
Deny	(1,1)	(−2, 2)
Confess	(2, −2)	(−1, −1)

The strategies available to the first player are listed along the far left side, and those of the second player are listed along the top. The payoff to each player is completely determined after a simultaneous choice of strategy by both players. They are listed as the elements in a vector, i.e., (payoff to the first player, payoff to the second player). For example, if the first player picks "Deny", and the second picks "Confess", then the vector in the matrix corresponding to this is $(-2, 2)$. So in this case, the first player has a payoff of -2 and the second player has a payoff of 2.

This particular payoff matrix corresponds to the following situation. A sheriff has two suspects who have committed a major crime, yet he has insufficient evidence to convict them of the major crime even though he does have evidence to convict them of a lesser offense. The two prisoners (player 1 and player 2) are interrogated separately. If both prisoners confess to the major crime (ie., they both chose the strategy "Confess"), then they are both punished severely. If they both deny, they are given mild sentences. However, if one confesses and the other denies, the one confessing receives no punishment, whereas the prisoner who denies is punished even more severely than if he confessed.

The traditional solution to this problem is through noticing the dominance in the matrix. That is, no matter what the "other" player does, each player is guaranteed a certain minimal payoff. Hence each player will be better off if he chooses the Confess strategy. This paradoxical result is brought about by the "greatest of all possible evils" bias of minimax theory. For even though both players would like to choose the deny strategy, they can be punished if the other player decides to confess (which would in fact be to the *other player's advantage*).

One attempt at a solution has been to allow the game to be repeated a finite number of times. We speculate that both players will choose the Deny strategy until the last move. Then, however, it is to the advantage of both players to Confess. Hence the next to the last game in effect becomes the last game. We eventually regress back to the first game, and the whole postulated argument collapses. This situation is described elegantly in Rapoport's *Two-Person Games*.

I would like to restate the problem to remove this impasse. Suppose that neither player knows when the last move will occur, and that players are interested solely in personal gain without regard to the consequences to the other player. We construct a meta-game by assuming that after a game is played, another game will be played with probability N, with $0 < N < 1$. The probability that there will be no next game is therefore $1 - N$. In previous discussions of prisoner's dilemma, N has had the values 0 or 1.

Call the first player to confess the First Player, and the other player the Second Player. It is convenient to assume that both players choose the Deny strategy first, and they continue choosing it. However once they choose the Confess strategy they keep choosing that strategy.

Let m be the game number in which the First Player has decided to make his first confession. Hence, he picks the Deny strategy for $m - 1$ games. Let n be the game number where the second player confesses first, with $m \leqq n$.

We want to evaluate the meta-game; first we note

$$\sum_{j=1}^{k} jN^j = \frac{N + N^{k+1}(N-1)k - N^{k+1}}{(N-1)^2}. \tag{1}$$

The value of the game to the First Player is the weighted summation of the expected value of each game. To find the value of the game we must add the expected values up to the point where the First Player "stabs the second in the back" + the sum while the Second Player waits to see what is going on + the sum while both players confess "to death". The value of the meta-game to the Second Player is similar, but *he loses points* (relative to the Confess strategy) while he waits during the second stage of the meta-game. This will produce asymmetric results if $m \neq n$.

Using (1), we see that the value of the first stage of the meta-game for the First Player is

$$\sum_{j=1}^{m-1} jN^{j-1} = \frac{1 + N^m m - N^m - mN^{m-1}}{(N-1)^2}. \tag{2}$$

The Second Player has nothing to gain during the second stage of the meta-game by waiting to confess. Therefore he will chose $n = m + 1$. Therefore, the expected value of the meta-game (for the First Player) in the second stage is

$$(m+1)N^{m-1}. \tag{3}$$

Finally, the expected value of the meta-game for the First Player during the third stage is

$$\begin{aligned}\sum_{j=1}^{\infty} (m+1-j)N^{m+j-1} &= (m+1)N^{m-1} \sum_{j=1}^{\infty} N^j - N^{m-1} \sum_{j=1}^{\infty} jN^j \\ &= \frac{(m+1)(1-N)N^m - N^m}{(N-1)^2}.\end{aligned} \tag{4}$$

Therefore, using (2), (3), and (4), we see that the value of the meta-game (V_m) is

$$V_m = \frac{1 + mN^m - N^m - mN^{m-1} + (m+1)(1-N)N^m - N^m}{(N-1)^2} + (m+1)N^{m-1}.$$

It is easy to simplify this expression to

$$V_m = \frac{1 - 3N^m + N^{m-1}}{(N-1)^2}. \tag{5}$$

For example, if $N = 1/2$ and $m = 4$, the value of the meta-game for the First Player is $V_4 = 1 + \frac{2}{2} + \frac{3}{4} + \frac{5}{8} + \frac{4}{16} + \frac{3}{32} + \cdots$.

The derived value from (5) is

$$V_4 = \frac{1 - \frac{3}{16} + \frac{1}{8}}{\frac{1}{4}} = \frac{15}{4}, \qquad \text{which checks.}$$

We want to know what constraints on N let V_m obtain its maximum value as m approaches infinity. In other words, when will the Deny strategy be chosen for all time. We want to solve

$$V_m < V_\infty. \tag{6}$$

By taking the limit of (5) as m goes to infinity, we rewrite (6) as

$$\frac{1 - 3N^m + N^{m-1}}{(N-1)^2} < \frac{1}{(N-1)^2}. \tag{7}$$

We solve (7) and find the condition $N > 1/3$.

From this it is clear that for N greater than 1/3 neither player wants to confess, and the dilemma is solved.

References

1. R. D. Luce and Howard Raiffa, Game Theory, Wiley, New York, 1958.
2. Guillermo Owen, Game Theory, Saunders, Philadelphia, 1968, pp. 139 and 140.
3. T. Parthasarathy and T. E. S. Raghavan, Some Topics in Two-Person Games, American Elsevier, New York, 1971, p. 152.
4. Anatol Rapoport, Two-Person Game Theory; The Essential Ideas, University of Michigan Press, Ann Arbor, 1966, pp, 128–144, especially p. 142.
5. ———, Prisoner's dilemma and other paradoxes, Scientific American, (July 1967) 50–56.

ON THE CRITICALLY DAMPED OSCILLATOR

RONALD S. BASLAW, Courant Institute, New York University and
HAROLD M. HASTINGS, SUNY at Binghamton

The usual basis for the solutions of the differential equation for the critically damped harmonic oscillator,

$$y'' - 2ry' + r^2y = 0, \quad r \text{ a constant} \tag{1}$$

Therefore, using (2), (3), and (4), we see that the value of the meta-game (V_m) is

$$V_m = \frac{1 + mN^m - N^m - mN^{m-1} + (m+1)(1-N)N^m - N^m}{(N-1)^2} + (m+1)N^{m-1}.$$

It is easy to simplify this expression to

$$V_m = \frac{1 - 3N^m + N^{m-1}}{(N-1)^2}. \tag{5}$$

For example, if $N = 1/2$ and $m = 4$, the value of the meta-game for the First Player is $V_4 = 1 + \frac{2}{2} + \frac{3}{4} + \frac{5}{8} + \frac{4}{16} + \frac{3}{32} + \cdots$.

The derived value from (5) is

$$V_4 = \frac{1 - \frac{3}{16} + \frac{1}{8}}{\frac{1}{4}} = \frac{15}{4}, \qquad \text{which checks.}$$

We want to know what constraints on N let V_m obtain its maximum value as m approaches infinity. In other words, when will the Deny strategy be chosen for all time. We want to solve

$$V_m < V_\infty. \tag{6}$$

By taking the limit of (5) as m goes to infinity, we rewrite (6) as

$$\frac{1 - 3N^m + N^{m-1}}{(N-1)^2} < \frac{1}{(N-1)^2}. \tag{7}$$

We solve (7) and find the condition $N > 1/3$.

From this it is clear that for N greater than 1/3 neither player wants to confess, and the dilemma is solved.

References

1. R. D. Luce and Howard Raiffa, Game Theory, Wiley, New York, 1958.
2. Guillermo Owen, Game Theory, Saunders, Philadelphia, 1968, pp. 139 and 140.
3. T. Parthasarathy and T. E. S. Raghavan, Some Topics in Two-Person Games, American Elsevier, New York, 1971, p. 152.
4. Anatol Rapoport, Two-Person Game Theory; The Essential Ideas, University of Michigan Press, Ann Arbor, 1966, pp, 128–144, especially p. 142.
5. ———, Prisoner's dilemma and other paradoxes, Scientific American, (July 1967) 50–56.

ON THE CRITICALLY DAMPED OSCILLATOR

RONALD S. BASLAW, Courant Institute, New York University and
HAROLD M. HASTINGS, SUNY at Binghamton

The usual basis for the solutions of the differential equation for the critically damped harmonic oscillator,

$$y'' - 2ry' + r^2y = 0, \quad r \text{ a constant} \tag{1}$$

consists of the functions $y = e^{rx}$ and $y = xe^{rx}$. The second solution is usually found by reduction of order (e.g., [**1**]) or some similar artifice. The main purpose of this brief note is to exhibit this solution, $y = xe^{rx}$, as the limit of a family of solutions to the family of differential equations parametrized by α:

$$\left\{y'' - 2ry' + \left(r^2 - \frac{\alpha}{4}\right) y = 0\right\}. \tag{2}$$

Equation (1) is obtained by setting $\alpha = 0$.

Secondly, we illustrate the following result, see, e.g., [**2**]. The solution to a normal linear differential equation (such as (2)) under fixed initial conditions depends continuously on the coefficients of that differential equation. This result is *not* used in our discussion of equation (2).

Observe that the function $y = xe^{rx}$ satisfies $y(0) = 0$, $y'(0) = 1$. Under these initial conditions, equations (2) have the solutions

$$y_\alpha = \frac{1}{\alpha}(e^{(r+\alpha/2)x} - e^{(r-\alpha/2)x}) \tag{3}$$

parametrized by α. Solutions (3) are easily obtained with the characteristic equation (e.g., [**1**]).

In order to compute the *pointwise* limit $\lim_{\alpha\to 0} y_\alpha(x)$, we observe that $e^{\alpha x/2}y_\alpha$ is the difference quotient used to compute $\partial/\partial s\ e^{sx}$ at $s = r$. Hence,

$$\lim_{\alpha\to 0} e^{\alpha x/2} y_\alpha(x) = \frac{\partial}{\partial s}(e^{sx})\Big|_{s=r} = xe^{rx}.$$

As α approaches 0, $e^{\alpha x/2}$ approaches 1; hence $y_\alpha(x)$ approaches $y(x) = xe^{rx}$ pointwise.

To obtain convergence uniform in x for x in a finite closed interval, first observe that

$$e^{\alpha x/2}y_\alpha - y = \frac{1}{\alpha}(e^{\alpha x} - 1 - \alpha x).$$

Since, for any fixed r, e^{rx} is a monotone function of x, and since $e^{\alpha x} - 1 - \alpha x$ is an increasing function of αx (take the derivative with respect to αx), estimates on $e^{\alpha x/2}y_\alpha - y$ uniform in x are easily obtained. Details are left to the reader. Since $e^{\alpha x/2}$ approaches 1 uniformly in x for x in a finite closed interval, $y = xe^{rx}$ is the *uniform limit* of $y_\alpha(x)$ on such an interval.

The reader may similarly study solutions to the family of equations parametrized by β:

$$\{y'' - 2(r + \beta)y' + r^2 y = 0\}.$$

References

1. W. Boyce and R. DiPrima, Elementary Differential Equations and Boundary Value Problems, 2nd ed., Wiley, New York, 1969.

2. G. Birkhoff and G. C. Rota, Ordinary Differential Equations, 2nd ed., Blaisdell, Waltham, 1969.

THE SUBGROUPS OF THE DIHEDRAL GROUP

STEPHAN R. CAVIOR, SUNY at Buffalo

Students in abstract algebra and number theory are usually interested to see that the arithmetic functions $\phi(n)$, Euler's function, and $d(n)$, the number of positive divisors of n, occur quite naturally in the solution of some group theory problems. It appears that the function $\sigma(n)$, the sum of the positive divisors of n, does also, in counting the number of subgroups of the dihedral group D_n. A formula is developed here for that number.

THEOREM. *The number of subgroups of the dihedral group* D_n $(n \geqq 3)$ *is* $d(n) + \sigma(n)$.

Proof. When considered geometrically, D_n consists of n rotations and n reflections of the regular n-gon. The subgroups of D_n are of two types: (1) Those containing rotations only, and (2) those containing rotations and reflections.

The subgroups of type 1 are simply the subgroups of Z_n, the cyclic group of order n, and the number of them is $d(n)$.

The subgroups of type 2 contain an equal number of rotations and reflections, say t, of each. Now the t rotations must comprise the unique subgroup of Z_n of order t, whence $t \mid n$, but the t reflections can be chosen in several ways. In fact, the axes of reflection form a star-shaped figure with equal central angles which can be positioned in the n-gon in n/t ways. Thus for each divisor t of n, there are n/t subgroups of type 2, and the total number of subgroups of type 2 must be

$$\sum_{t \mid n} n/t = \sum_{t \mid n} t = \sigma(n).$$

This completes the proof of the theorem.

It is of interest to note that when p is a prime greater than 2, the number of subgroups of D_p is simply $p + 3$.

ON THE PROOF THAT ALL EVEN PERFECT NUMBERS ARE OF EUCLID'S TYPE

WAYNE L. McDANIEL, University of Missouri—St. Louis

In 1961, Spira [5] defined the sum of divisors function and the notion of perfect number in the ring G of Gaussian integers. Spira proved an analog in G of Euclid's theorem that every number of the form $2^{p-1}(2^p - 1)$, where p and $2^p - 1$ are prime, is a perfect number, and commented that the proofs of Euler's converse of the theorem do not appear to generalize to the complex case.

The standard proofs of Euler's converse are of course quite elementary; it may, nevertheless, be of interest to note that there does exist a simple proof of the converse

THE SUBGROUPS OF THE DIHEDRAL GROUP

STEPHAN R. CAVIOR, SUNY at Buffalo

Students in abstract algebra and number theory are usually interested to see that the arithmetic functions $\phi(n)$, Euler's function, and $d(n)$, the number of positive divisors of n, occur quite naturally in the solution of some group theory problems. It appears that the function $\sigma(n)$, the sum of the positive divisors of n, does also, in counting the number of subgroups of the dihedral group D_n. A formula is developed here for that number.

THEOREM. *The number of subgroups of the dihedral group* D_n $(n \geqq 3)$ *is* $d(n) + \sigma(n)$.

Proof. When considered geometrically, D_n consists of n rotations and n reflections of the regular n-gon. The subgroups of D_n are of two types: (1) Those containing rotations only, and (2) those containing rotations and reflections.

The subgroups of type 1 are simply the subgroups of Z_n, the cyclic group of order n, and the number of them is $d(n)$.

The subgroups of type 2 contain an equal number of rotations and reflections, say t, of each. Now the t rotations must comprise the unique subgroup of Z_n of order t, whence $t \mid n$, but the t reflections can be chosen in several ways. In fact, the axes of reflection form a star-shaped figure with equal central angles which can be positioned in the n-gon in n/t ways. Thus for each divisor t of n, there are n/t subgroups of type 2, and the total number of subgroups of type 2 must be

$$\sum_{t \mid n} n/t = \sum_{t \mid n} t = \sigma(n).$$

This completes the proof of the theorem.

It is of interest to note that when p is a prime greater than 2, the number of subgroups of D_p is simply $p + 3$.

ON THE PROOF THAT ALL EVEN PERFECT NUMBERS ARE OF EUCLID'S TYPE

WAYNE L. McDANIEL, University of Missouri—St. Louis

In 1961, Spira [5] defined the sum of divisors function and the notion of perfect number in the ring G of Gaussian integers. Spira proved an analog in G of Euclid's theorem that every number of the form $2^{p-1}(2^p - 1)$, where p and $2^p - 1$ are prime, is a perfect number, and commented that the proofs of Euler's converse of the theorem do not appear to generalize to the complex case.

The standard proofs of Euler's converse are of course quite elementary; it may, nevertheless, be of interest to note that there does exist a simple proof of the converse which may be generalized to the complex case (a proof of an analog in G of the converse appears in Acta Arithmetica [4]).

THEOREM (Euler's Converse). *If n is an even perfect number, there exist prime numbers p and $2^p - 1$ such that $n = 2^{p-1}(2^p - 1)$.*

Our proof differs from Euler's original proof (see [2] p. 88, or [3]) and from the variation of Euler's proof (apparently due to Dickson [1]) which appears in most introductory number theory books in one essential respect: it focuses attention on the least prime factor of $2^k - 1$ rather than upon a multiple $c(2^k - 1)$ of $2^k - 1$ (we conclude that the least prime factor is $2^k - 1$ and the usual proof concludes that $c = 1$). It is precisely this change in point of view which permits the generalization to the ring of Gaussian integers.

We use the facts that the sum of divisors function σ is multiplicative, and that if q is prime, $a > 1$ and $(q, a) = 1$, then

$$\frac{\sigma(q^\alpha a)}{q^\alpha a} = \frac{\sigma(q^\alpha)\sigma(a)}{q^\alpha a} > \frac{\sigma(q^\alpha)}{q^\alpha} = \frac{1 + q + \cdots + q^\alpha}{q^\alpha} \geqq \frac{q+1}{q}.$$

Proof of the Theorem. Suppose $n = 2^{k-1}m$, m odd, is an even perfect number. Since

$$2^k m = 2n = \sigma(n) = \sigma(2^{k-1})\sigma(m) = (2^k - 1)\sigma(m),$$

every prime divisor of $2^k - 1$ divides m. Let q be the least prime divisor of $2^k - 1$ and α be the largest integer such that q^α divides m.

Now,

$$1 = \frac{\sigma(n)}{2n} = \frac{\sigma(2^{k-1})\sigma(m)}{2^k m} \geqq \frac{\sigma(2^{k-1})\sigma(q^\alpha)}{2^k q^\alpha}$$

$$\geqq \frac{(2^k - 1)(q + 1)}{2^k q} = 1 + \frac{(2^k - 1) - q}{2^k q}$$

holds only if $q = 2^k - 1$, $\alpha = 1$ and $m = q$; hence $n = 2^{k-1}(2^k - 1)$ and $2^k - 1$ is prime.

It may be worth noting that this easy proof, in addition to being generalizable to the complex case, shares with Euler's original proof (as opposed to the proof found in nearly all introductory number theory textbooks) an additional pedagogical plus: it examines the function $\sigma(n)/n$, and this is the basic approach used in the study of *odd* perfect numbers.

References

1. L. E. Dickson, Introduction to the Theory of Numbers, University of Chicago Press, 1929.
2. L. Euler, Opera Postuma 1, Eggers et Socios, Petropoli (Leningrad), 1862.
3. G. H. Hardy and E. M. Wright, An Introduction to the Theory of Numbers, 4th ed., Clarendon Press, Oxford, 1960.
4. W. L. McDaniel, Perfect Gaussian integers, Acta Arith., 25 (1973) 137–144.
5. R. Spira, The complex sum of divisors, Amer. Math. Monthly, 68 (1961) 120–124.

A NOTE ON SUMS OF THREE SQUARES IN $GF[q, x]$

L. CARLITZ, Duke University

Let $F = GF(q)$ denote the finite field of *odd* order q. Let a, b, c, d denote four arbitrary nonzero elements of F. Let M denote an arbitrary polynomial in $F[x]$. Eckford Cohen ([**2**], see also [**1**]) has determined the number of solutions A, B, C, D of the equation

$$M = aA^2 + bB^2 + cC^2 + dD^2, \tag{1}$$

where A, B, C, D are polynomials in $F[x]$ of degree $\leqq k$, where $k > \deg M$. It follows from this result that (1) is always solvable with A, B, C, D unrestricted.

On the other hand, it is evident that the equation

$$M = aA^2 + bB^2 \tag{2}$$

is not solvable for M of odd degree and $-ab$ not a square in F. Thus it is of interest to know whether the equation

$$M = aA^2 + bB^2 + cC^2 \tag{3}$$

is always solvable.

We shall prove the following result:

THEOREM 1. *Let a, b, c be arbitrary nonzero elements of F and let M be an arbitrary polynomial in $F[x]$. Then the equation* (3) *is always solvable with A, B, C in $F[x]$.*

Proof. Consider the equation

$$x = a(rx + r')^2 + b(sx + s')^2 + c(tx + t')^2 \tag{4}$$

where $r, r', s, s', t, t' \in F$. This is equivalent to the system of equations

$$\begin{cases} ar^2 + bs^2 + ct^2 = ar'^2 + bs'^2 + ct'^2 = 0 \\ 2arr' + 2bss' + 2ctt' = 1\,. \end{cases} \tag{5}$$

It is well known (and easy to prove) that the equation

$$ar^2 + bs^2 + ct^2 = 0 \tag{6}$$

has nontrivial solutions in F. We may assume that r, s, t is a solution of (5) with $r \neq 0$, $s \neq 0$. Put $r_1 = r$, $s_1 = -s$, $t_1 = t$, so that

$$rr_1 + ss_1 + tt_1 = r^2 - s^2 + t^2 = -2s^2 \neq 0\,.$$

Hence if we take

$$r' = \frac{r}{2s^2}, \quad s' = -\frac{1}{2s}, \quad t' = \frac{t}{2s^2},$$

it is clear that (5) is satisfied.

Now replace x by M in (4) and the theorem follows at once.

We may evidently state the following more general result:

THEOREM 2. *Let a, b, c be arbitrary nonzero elements of F and let M be an arbitrary polynomial in $F[x_1,\cdots,x_n]$. Then the equation* (3) *is always solvable with A, B, C in $F[x_1,\cdots,x_n]$.*

It would of course be interesting to prove these results with bound on the degrees of A, B, C.

Supported in part by NSF grant GP-37924.

References

1. L. Carlitz, Representations of arithmetic functions in *GF* $[p^n, x]$, Duke Math. J., 14 (1947) 1121–1137.

2. Eckford Cohen, Sums of an even number of squares in *GF* $[p^n, x]$ II, this MAGAZINE, 14 (1947) 543–557.

CONVEX BODIES AND LATTICE POINTS

P. R. SCOTT, University of Adelaide

1. Introduction. A number of results are known about convex bodies in E^n and their relation to the points of the integral lattice. The best known of these is Minkowski's Theorem:

THEOREM 1. *If $\mathscr{K}$ is a closed convex body in E^n, symmetric about the origin O, and having volume $V(\mathscr{K}) \geqq 2^n$, then $\mathscr{K}$ contains a nonzero point of the integral lattice.*

Recently it has been shown [3] that Minkowski's Theorem continues to hold for nonsymmetric convex sets $\mathscr{K}$ in E^2, providing there exists a chord of $\mathscr{K}$ with midpoint O which partitions $\mathscr{K}$ into two disjoint regions of equal area. This result is generalized in section 2.

The following result for convex sets in E^3 which are not necessarily symmetric was proved by Ehrhart [2]. (See [1] for the corresponding result in the plane.)

THEOREM 2. *If $\mathscr{K}$ is a closed convex solid of revolution with center of gravity at the origin O, and volume $V(\mathscr{K}) \geqq \frac{4^4}{3^3}\left(= 9\frac{13}{27}\right)$, then $\mathscr{K}$ contains a nonzero point of the integral lattice.*

We extend this result in section 3.

2. An extension of Minkowski's theorem. Let $\mathscr{K}$ be a closed convex body in E^n, and

Now replace x by M in (4) and the theorem follows at once.

We may evidently state the following more general result:

THEOREM 2. *Let a, b, c be arbitrary nonzero elements of F and let M be an arbitrary polynomial in $F[x_1, \cdots, x_n]$. Then the equation* (3) *is always solvable with A, B, C in $F[x_1, \cdots, x_n]$.*

It would of course be interesting to prove these results with bound on the degrees of A, B, C.

Supported in part by NSF grant GP-37924.

References

1. L. Carlitz, Representations of arithmetic functions in $GF[p^n, x]$, Duke Math. J., 14 (1947) 1121–1137.

2. Eckford Cohen, Sums of an even number of squares in $GF[p^n, x]$ II, this MAGAZINE, 14 (1947) 543–557.

CONVEX BODIES AND LATTICE POINTS

P. R. SCOTT, University of Adelaide

1. Introduction. A number of results are known about convex bodies in E^n and their relation to the points of the integral lattice. The best known of these is Minkowski's Theorem:

THEOREM 1. *If $\mathscr{K}$ is a closed convex body in E^n, symmetric about the origin O, and having volume $V(\mathscr{K}) \geqq 2^n$, then $\mathscr{K}$ contains a nonzero point of the integral lattice.*

Recently it has been shown [3] that Minkowski's Theorem continues to hold for nonsymmetric convex sets $\mathscr{K}$ in E^2, providing there exists a chord of $\mathscr{K}$ with midpoint O which partitions $\mathscr{K}$ into two disjoint regions of equal area. This result is generalized in section 2.

The following result for convex sets in E^3 which are not necessarily symmetric was proved by Ehrhart [2]. (See [1] for the corresponding result in the plane.)

THEOREM 2. *If $\mathscr{K}$ is a closed convex solid of revolution with center of gravity at the origin O, and volume $V(\mathscr{K}) \geqq \frac{4^4}{3^3}\left(= 9\frac{13}{27}\right)$, then $\mathscr{K}$ contains a nonzero point of the integral lattice.*

We extend this result in section 3.

2. An extension of Minkowski's theorem. Let $\mathscr{K}$ be a closed convex body in E^n, and

π an intersecting hyperplane. We say that the section $\mathscr{K} \cap \pi$ is *visible* if there is some point P exterior to $\mathscr{K}$ (perhaps 'at infinity') from which every boundary point of $\mathscr{K} \cap \pi$ (relative to π) is visible. We call such a point P a *point of visibility*.

Further, if π partitions E^n into the two halfspaces π^+ and π^-, and if $V(\pi^+ \cap \mathscr{K}) > V(\pi^- \cap \mathscr{K})$, then π^+ is the *heavy side* of π (relative to $\mathscr{K}$). If $V(\pi^+ \cap \mathscr{K}) = V(\pi^- \cap \mathscr{K})$, then both π^+ and π^- are heavy.

THEOREM 3. *Let $\mathscr{K}$ be a closed convex body in E^n containing the origin O, and let π be a plane through O such that $\pi \cap \mathscr{K}$ is symmetric about O, and $\pi \cap \mathscr{K}$ is visible from the heavy side of π. Then if $V(\mathscr{K}) \geqq 2^n$, $\mathscr{K}$ contains a nonzero point of the integral lattice.*

Proof. Let π partition $\mathscr{K}$ into the sets $\mathscr{K}_1$, $\mathscr{K}_2$, and suppose that a point of visibility V lies on the $\mathscr{K}_1$-side of π. Let $\mathscr{K}_1'$ denote the reflection of $\mathscr{K}_1$ in the origin O, and set $\mathscr{K}^* = \mathscr{K}_1 \cup \mathscr{K}_1'$. By construction, $\mathscr{K}^*$ is symmetric about the origin, and $V(\mathscr{K}^*) \geqq 2^n$.

Let B be a boundary point of $\mathscr{K}^*$. To show that $\mathscr{K}^*$ is convex, it is sufficient to show that for each such point B, there is a hyperplane σ which supports $\mathscr{K}^*$ 'locally' (that is, if $N(B,\varepsilon)$ is a spherical ε-neighborhood of B ($\varepsilon > 0$), then σ supports $\mathscr{K}^* \cap N(B,\varepsilon)$; see for example, [4]).

If $B \notin \pi$, such a hyperplane clearly exists, since $\mathscr{K}, \mathscr{K}'$ are convex. Suppose then that $B \in \pi$. Let V' denote the reflection of V in O, and consider the double-cone $\mathscr{C}$ having (common) base $\mathscr{K}^* \cap \pi$ and vertices V, V'. Each half of $\mathscr{C}$ is convex, and since VV' meets $\mathscr{C} \cap \pi$ in O, $\mathscr{C}$ is itself convex. Hence at B there exists a hyperplane of support to $\mathscr{C}$. Since $\mathscr{K}^* \subset \mathscr{C}$, this hyperplane also supports $\mathscr{K}^*$ as required. Hence $\mathscr{K}^*$ is convex.

But now by Minkowski's theorem, $\mathscr{K}^*$ contains a nonzero point of the integral lattice. Using the symmetry of $\mathscr{K}^*$, we deduce that $\mathscr{K}_1$, and so $\mathscr{K}$, contains a nonzero point of the integral lattice.

3. An extension of Ehrhart's theorem.

THEOREM 4. *Let $\mathscr{K}$ be a closed convex body in E^3 containing the origin, and suppose there exists a plane π which meets $\mathscr{K}$ in a section which is visible and symmetric about O, and which contains the centre of gravity of $\mathscr{K}$. Then if $V(\mathscr{K}) \geqq \frac{4^4}{3^3}$, $\mathscr{K}$ contains a nonzero point of the integral lattice.*

Proof. Let π partition $\mathscr{K}$ into the two sets $\mathscr{K}_1$, $\mathscr{K}_2$, and suppose that a point of visibility lies on the $\mathscr{K}_1$-side of π. Let $\mathscr{K}_1'$ denote the reflection of $\mathscr{K}_1$ in the origin, and set $\mathscr{K}^* = \mathscr{K}_1 \cup \mathscr{K}_1'$. As in Theorem 3, $\mathscr{K}^*$ is convex and symmetric about the origin. Also, since π passes through the center of gravity of $\mathscr{K}$, it is known that $V(\mathscr{K}_1) \geqq \frac{27}{64} V(\mathscr{K})$ (see [2]). Hence using the hypothesis of the theorem, $V(\mathscr{K}_1) \geqq 4$ and $V(\mathscr{K}^*) \geqq 8$. Applying Minkowski's theorem to $\mathscr{K}^*$ we deduce that $\mathscr{K}^*$ contains a nonzero point of the integral lattice.

In [1], Ehrhart presumes that his theorem generalizes to E^n, but gives no proof. If, as seems likely, his theorem is valid in E^n, the generalization of Theorem 4 will also be valid, and will read:

Let $\mathscr{K}$ be a closed convex body in E^n containing the origin, and suppose there exists a hyperplane π which meets $\mathscr{K}$ in a section which is visible and symmetric about O, and which contains the center of gravity of $\mathscr{K}$. Then if

$$V(\mathscr{K}) \geqq \frac{2^{n-1}(n+1)^n}{n^n},$$

$\mathscr{K}$ contains a nonzero point of the integral lattice.

4. A concluding problem. It is clear that there are many closed convex bodies $\mathscr{K}$ which satisfy the conditions of Theorem 3, but which are not symmetric about O. On the other hand, it is not clear whether every closed convex body $\mathscr{K}$ which is symmetric about O has a section which is visible and symmetric about O. For example the regular dodecahedron is a centrally symmetric body, yet not every section through the center is visible.

References

1. E. Ehrhart, Une Généralisation du théorème de Minkowski, Comptes Rendus, 240 (1955) 483–485.
2. ———, Sur les ovales et les ovoïdes, Comptes Rendus, 240 (1955) 583–585.
3. P. R. Scott, On Minkowski's theorem, this MAGAZINE, 47 (1974) 277.
4. F. A. Valentine, Convex Sets, McGraw-Hill, New York, 1964.

PRODUCTS OF SUMS OF POWERS

MELVYN B. NATHANSON, The Institute for Advanced Study, Princeton

Let m and n be positive integers, and let $x_1, x_2, \cdots, x_m$, $y_1, y_2, \cdots, y_m$ be indeterminates. We say that $H(m,n)$ holds if there exist polynomials $z_1, z_2, \cdots, z_m$ with integer coefficients in the $2m$ variables $x_1, \cdots, x_m, y_1, \cdots, y_m$ such that

$$(*) \qquad (x_1^n + x_2^n + \cdots + x_m^n)(y_1^n + y_2^n + \cdots + y_m^n) = (z_1^n + z_2^n + \cdots + z_m^n).$$

Clearly, $H(1,n)$ and $H(m,1)$ hold for all m and n.

In 1923, Hürwitz [1] proved that $H(m,2)$ holds if and only if $m = 1,2,4,8$. These identities follow from the multiplication rules for the real, complex, quaternion, and Cayley numbers. Here is the simplest proof that $H(3,2)$ is impossible: $2 \cdot 14 = 28$. That is, both 2 and 14 are sums of three squares. If there existed a polynomial identity (*) with $m = 3$ and $n = 2$, then also 28 would be a sum of three squares. But it is not. We can apply this simple arithmetic technique to show that other identities of the form (*) cannot hold.

In [1], Ehrhart presumes that his theorem generalizes to E^n, but gives no proof. If, as seems likely, his theorem is valid in E^n, the generalization of Theorem 4 will also be valid, and will read:

Let $\mathscr{K}$ be a closed convex body in E^n containing the origin, and suppose there exists a hyperplane π which meets $\mathscr{K}$ in a section which is visible and symmetric about O, and which contains the center of gravity of $\mathscr{K}$. Then if

$$V(\mathscr{K}) \geqq \frac{2^{n-1}(n+1)^n}{n^n},$$

$\mathscr{K}$ contains a nonzero point of the integral lattice.

4. A concluding problem. It is clear that there are many closed convex bodies $\mathscr{K}$ which satisfy the conditions of Theorem 3, but which are not symmetric about O. On the other hand, it is not clear whether every closed convex body $\mathscr{K}$ which is symmetric about O has a section which is visible and symmetric about O. For example the regular dodecahedron is a centrally symmetric body, yet not every section through the center is visible.

References

1. E. Ehrhart, Une Généralisation du théorème de Minkowski, Comptes Rendus, 240 (1955) 483–485.
2. ———, Sur les ovales et les ovoïdes, Comptes Rendus, 240 (1955) 583–585.
3. P. R. Scott, On Minkowski's theorem, this MAGAZINE, 47 (1974) 277.
4. F. A. Valentine, Convex Sets, McGraw-Hill, New York, 1964.

PRODUCTS OF SUMS OF POWERS

MELVYN B. NATHANSON, The Institute for Advanced Study, Princeton

Let m and n be positive integers, and let $x_1, x_2, \cdots, x_m$, $y_1, y_2, \cdots, y_m$ be indeterminates. We say that $H(m,n)$ holds if there exist polynomials $z_1, z_2, \cdots, z_m$ with integer coefficients in the $2m$ variables $x_1, \cdots, x_m, y_1, \cdots, y_m$ such that

$$(*) \qquad (x_1^n + x_2^n + \cdots + x_m^n)(y_1^n + y_2^n + \cdots + y_m^n) = (z_1^n + z_2^n + \cdots + z_m^n).$$

Clearly, $H(1,n)$ and $H(m,1)$ hold for all m and n.

In 1923, Hürwitz [1] proved that $H(m,2)$ holds if and only if $m = 1,2,4,8$. These identities follow from the multiplication rules for the real, complex, quaternion, and Cayley numbers. Here is the simplest proof that $H(3,2)$ is impossible: $2 \cdot 14 = 28$. That is, both 2 and 14 are sums of three squares. If there existed a polynomial identity (*) with $m = 3$ and $n = 2$, then also 28 would be a sum of three squares. But it is not. We can apply this simple arithmetic technique to show that other identities of the form (*) cannot hold.

THEOREM 1. *$H(3,3)$ does not hold.*

Proof. If there exists an identity (*) with $m = n = 3$, then $2\cdot 2 = 4$ is a sum of three cubes. But any cube is congruent to 0, 1, or 8 modulo 9, and so no number congruent to 4 modulo 9 is a sum of three cubes. Therefore, $H(3,3)$ is impossible.

THEOREM 2. *$H(2,n)$ does not hold for any $n > 2$.*

Proof. If there existed polynomials z_1 and z_2 such that $(x_1^n + x_2^n)(y_1^n + y_2^n) = z_1^n + z_2^n$, then $(1^n + 1^n)(1^n + 1^n) = 4$ would be a sum or difference of nth powers for some $n > 2$. But this is impossible.

THEOREM 3. *Let $m > 2$. Then $H(m,2n)$ holds for only finitely many n.*

Proof. Suppose that $H(m,2n)$ holds, and set $x_1 = x_2 = \cdots = x_m = y_1 = y_2 = \cdots = y_m = 1$. Then m^2 is a sum of m $(2n)$th powers. At least one of these powers must be at least 2 in absolute value. That is, $2^{2n} \leqq m^2$, or $n \leqq \log m/\log 2$.

One can conjecture, for any m, that $H(m,n)$ holds for only finitely many n. More generally, one can conjecture that $H(m,n)$ is impossible for any $m > 1$ and $n > 2$.

Reference

1. A. Hürwitz, Über die Komposition der quadratischen Formen, Math. Ann., 88 (1923) 1–25.

BOOK REVIEWS

EDITED BY ADA PELUSO AND WILLIAM WOOTON

Materials intended for review should be sent to: Professor Ada Peluso, Department of Mathematics, Hunter College of CUNY, 695 Park Avenue, New York, New York 10021, or to Professor William Wooton, 1495 La Linda Drive, Lake San Marcos, California 92069. A boldface capital C in the margin indicates that a review is based in part on classroom use.

Mathematics for the Biological Sciences. By Stanley I. Grossman and James E. Turner. Macmillan, New York, 1974. xi + 512 pp. $10.95.

In the past few years we have witnessed an appreciable increase in the number of texts on elementary mathematics written for students of the biological and life sciences. The present volume is designed to meet the needs of a variety of such students. The book contains nine chapters, six appendices, and six tables. The first six chapters, which comprise approximately seventy-five percent of the text material, are devoted to finite mathematics. This includes the usual material on discrete probability, matrix algebra, linear programming, markov chains, game theory, and difference equations. The authors have incorporated a large number of illustrative examples and exercises on the uses of finite mathematics in the life sciences, which

THEOREM 1. *$H(3,3)$ does not hold.*

Proof. If there exists an identity (*) with $m = n = 3$, then $2\cdot 2 = 4$ is a sum of three cubes. But any cube is congruent to 0, 1, or 8 modulo 9, and so no number congruent to 4 modulo 9 is a sum of three cubes. Therefore, $H(3,3)$ is impossible.

THEOREM 2. *$H(2,n)$ does not hold for any $n > 2$.*

Proof. If there existed polynomials z_1 and z_2 such that $(x_1^n + x_2^n)(y_1^n + y_2^n) = z_1^n + z_2^n$, then $(1^n + 1^n)(1^n + 1^n) = 4$ would be a sum or difference of nth powers for some $n > 2$. But this is impossible.

THEOREM 3. *Let $m > 2$. Then $H(m,2n)$ holds for only finitely many n.*

Proof. Suppose that $H(m,2n)$ holds, and set $x_1 = x_2 = \cdots = x_m = y_1 = y_2 = \cdots = y_m = 1$. Then m^2 is a sum of m $(2n)$th powers. At least one of these powers must be at least 2 in absolute value. That is, $2^{2n} \leqq m^2$, or $n \leqq \log m/\log 2$.

One can conjecture, for any m, that $H(m,n)$ holds for only finitely many n. More generally, one can conjecture that $H(m,n)$ is impossible for any $m > 1$ and $n > 2$.

Reference

1. A. Hürwitz, Über die Komposition der quadratischen Formen, Math. Ann., 88 (1923) 1–25.

BOOK REVIEWS

EDITED BY ADA PELUSO AND WILLIAM WOOTON

Materials intended for review should be sent to: Professor Ada Peluso, Department of Mathematics, Hunter College of CUNY, 695 Park Avenue, New York, New York 10021, or to Professor William Wooton, 1495 La Linda Drive, Lake San Marcos, California 92069. A boldface capital C in the margin indicates that a review is based in part on classroom use.

Mathematics for the Biological Sciences. By Stanley I. Grossman and James E. Turner. Macmillan, New York, 1974. xi + 512 pp. $10.95.

In the past few years we have witnessed an appreciable increase in the number of texts on elementary mathematics written for students of the biological and life sciences. The present volume is designed to meet the needs of a variety of such students. The book contains nine chapters, six appendices, and six tables. The first six chapters, which comprise approximately seventy-five percent of the text material, are devoted to finite mathematics. This includes the usual material on discrete probability, matrix algebra, linear programming, markov chains, game theory, and difference equations. The authors have incorporated a large number of illustrative examples and exercises on the uses of finite mathematics in the life sciences, which makes this book particularly appealing. A precalculus student in the life sciences (and others too) can certainly benefit from a course based on these six chapters.

The last three chapters consist of an introduction to differential equations, continuous probability, and modelling in biology. This material presupposes some knowledge of calculus of a single variable, which is summarized in the appendices. Nearly all the examples on uses of differential equations have been drawn from population dynamics. Compartmental analysis and other physiological applications were not mentioned.

The authors stated in their preface that it is their aim "to make the relevant mathematics accessible in a reasonable time and to develop the student's ability to relate mathematics to problems in biology and medicine." The variety of mathematical tools needed in these disciplines today is simply too large to be encompassed in any single text so an author must select those topics and examples which he/she believes will have the widest appeal. On the whole this reviewer believes the authors have succeeded in their chosen task quite well.

P. K. WONG, Michigan State University

ANNOUNCEMENT OF LESTER R. FORD AWARDS

At its meeting on January 27, 1965, in Denver, Colorado, the Board of Governors authorized a number of awards, to be named after Lester R. Ford, Sr., to authors of expository articles published in the MONTHLY and the MATHEMATICS MAGAZINE. A maximum of six awards will be made annually; each award is in the amount of $100. The articles are to be selected by a subcommittee of the Committee on Publications appointed for this purpose.

The 1974 recipients of these Awards, selected by a committee consisting of E. F. Beckenbach, Chairman, Emil Grosswald, and D. E. Richmond, were announced by President R. P. Boas at the business meeting of the Association on January 26, 1975, in Washington, D. C. The recipients of the Ford Awards for articles published in 1973 were the following:

Patrick Billingsley, Prime Numbers and Brownian Motion, MONTHLY, 80 (1973) 1099–1115.

Garrett Birkhoff, Current Trends in Algebra, MONTHLY, 80 (1973) 760–782.

Martin Davis, Hilbert's Tenth Problem is Unsolvable, MONTHLY, 80 (1973) 233–269.

I. J. Schoenberg, The Elementary Cases of Landau's Problem of Inequalities between Derivatives, MONTHLY, 80 (1973) 121–158.

L. A. Steen, Highlights in the History of Spectral Theory, MONTHLY, 80 (1973) 359–381.

R. J. Wilson, An Introduction to Matroid Theory, MONTHLY, 80 (1973) 500–525.

HENRY L. ALDER, *Secretary*

makes this book particularly appealing. A precalculus student in the life sciences (and others too) can certainly benefit from a course based on these six chapters.

The last three chapters consist of an introduction to differential equations, continuous probability, and modelling in biology. This material presupposes some knowledge of calculus of a single variable, which is summarized in the appendices. Nearly all the examples on uses of differential equations have been drawn from population dynamics. Compartmental analysis and other physiological applications were not mentioned.

The authors stated in their preface that it is their aim "to make the relevant mathematics accessible in a reasonable time and to develop the student's ability to relate mathematics to problems in biology and medicine." The variety of mathematical tools needed in these disciplines today is simply too large to be encompassed in any single text so an author must select those topics and examples which he/she believes will have the widest appeal. On the whole this reviewer believes the authors have succeeded in their chosen task quite well.

P. K. WONG, Michigan State University

ANNOUNCEMENT OF LESTER R. FORD AWARDS

At its meeting on January 27, 1965, in Denver, Colorado, the Board of Governors authorized a number of awards, to be named after Lester R. Ford, Sr., to authors of expository articles published in the MONTHLY and the MATHEMATICS MAGAZINE. A maximum of six awards will be made annually; each award is in the amount of $100. The articles are to be selected by a subcommittee of the Committee on Publications appointed for this purpose.

The 1974 recipients of these Awards, selected by a committee consisting of E. F. Beckenbach, Chairman, Emil Grosswald, and D. E. Richmond, were announced by President R. P. Boas at the business meeting of the Association on January 26, 1975, in Washington, D. C. The recipients of the Ford Awards for articles published in 1973 were the following:

Patrick Billingsley, Prime Numbers and Brownian Motion, MONTHLY, 80 (1973) 1099–1115.

Garrett Birkhoff, Current Trends in Algebra, MONTHLY, 80 (1973) 760–782.

Martin Davis, Hilbert's Tenth Problem is Unsolvable, MONTHLY, 80 (1973) 233–269.

I. J. Schoenberg, The Elementary Cases of Landau's Problem of Inequalities between Derivatives, MONTHLY, 80 (1973) 121–158.

L. A. Steen, Highlights in the History of Spectral Theory, MONTHLY, 80 (1973) 359–381.

R. J. Wilson, An Introduction to Matroid Theory, MONTHLY, 80 (1973) 500–525.

HENRY L. ALDER, *Secretary*

PROBLEMS AND SOLUTIONS

Readers of this department are invited to submit for solution problems believed to be new that may arise in study, in research, or in extra-academic situations. Problems may be submitted from any branch of mathematics and ranging in subject content from that accessible to the talented high school student to problems challenging to the professional mathematician. Proposals should be accompanied by solutions, when available, and by any information that will assist the editor. Ordinarily, problems in well-known textbooks should not be submitted.

The asterisk (*) will be placed by the problem number to indicate that the proposer did not supply a solution. Readers' solutions are solicited for all problems. Proposers' solutions may not be "best possible" and solutions by others will be given preference.

Solutions should be submitted on separate, signed sheets. Figures should be drawn in India ink, and exactly the size desired for reproduction.

Send all communications for this department to Dan Eustice, the Ohio State University, 231 W. 18th Ave., Columbus, Ohio 43210.

To be considered for publication, solutions should be mailed before October 1, 1975.

PROPOSALS

929. *Proposed by Charles W. Trigg, San Diego, California.*

Show that there are only two octahedrons with equilateral triangular faces.

930. *Proposed by M. S. Klamkin, University of Waterloo.*

Solve the system of equations $(x_i - a_{i+1})(x_{i+1} - a_{i+3}) = a_{i+2}^2$, $i = 1, 2, \cdots, n$, for the x_i's, where $a_{n+i} = a_i$, $x_{n+i} = x_i$, and $a_1 a_2 \cdots a_n \neq 0$.

931. *Proposed by Alan Wayne, Holiday, Florida.*

For each $r \leqq n$ in a list of n statements, the rth statement is: "The number of false statements in this list is greater than r." Determine the truth value of each statement.

932. *Proposed by R. A. Struble, N. C. State University at Raleigh.*

Is there a topology for the set of real n-tuples other than the Euclidean topology, relative to which the family of connected sets is exactly the usual one?

933. *Proposed by Norman Schaumberger, Bronx Community College.*

Show that

$$\lim_{n\to\infty} \frac{n^2}{(1 \cdot 2^2 \cdot 3^3 \cdots n^n)^{4/n^2}} = e.$$

934. *Proposed by Erwin Just, Bronx Community College.*

From the first kn positive integers, choose a subset, K, consisting of $(k-1)n+1$ distinct integers. Prove that at least one member of K is the sum of k members (not necessarily distinct) of K.

935. *Proposed by Qazi Zameeruddin, K. M. College, Delhi–7, India.*

It is well known that the additive group Q of the rational numbers has no maximal subgroup. Is this statement true for the multiplicative group Q^* of nonzero rational numbers? If the answer is no, then characterize all maximal subgroups of Q^*.

936.* *Proposed by Jack Garfunkel, Flushing, New York.*

It is known that $h_a + h_b + h_c \leqq \sqrt{3}s$, where the h's represent altitudes to sides a, b, and c and s represents the semiperimeter of triangle ABC. Prove or disprove the stronger inequality $t_a + t_b + m_c \leqq \sqrt{3}s$, where the t's are the angle bisectors and m_c is the median to side c.

QUICKIES

From time to time this department will publish problems which may be solved by laborious methods, but which with the proper insight may be disposed of with dispatch. Readers are urged to submit their favorite problems of this type, together with the elegant solutions and the source, if known.

Q614. Find the sum of all distinct positive divisors of 104, 060, 401.

(*Note.* Although the solution is easy, a number of competent people do not get it right away. Try it first before you look at the solution.)

[*Submitted by Rod Cooper*]

Q615. Let q be any positive integer except an integral power of 10. Let 10^a be the integral power of 10 satisfying the inequality $10^a > q > 10^{a-1}$. Expand $1/q$ as the sum of an infinite geometric series whose first term and ratio depend on only q and 10^a.

[*Submitted by Joseph A. Wehlen*]

Q616. The faces of a tetrahedron and a hexahedron (triangular dipyramid) are congruent equilateral triangles. What is the ratio of the radii of their inscribed spheres?

[*Submitted by C. W. Trigg*]

Q617. If a and b are positive real numbers, show that for any positive integers m and n there is always a rational number of the form x^m/y^n between a and b with x and y integers.

[*Submitted by Norman Schaumberger and Erwin Just*]

Q618. If $1 \geqq x, y, z \geqq -1$, show that

$$\frac{1}{(1-x)(1-y)(1-z)} + \frac{1}{(1+x)(1+y)(1+z)} \geqq 2$$

with equality iff $x = y = z = 0$.

[*Submitted by M. S. Klamkin*]

Q619. Using "the distributivity of addition over multiplication", Lucky Larry obtained the correct answer to $(0.5) + (0.2)(0.3)$ by multiplying 0.7 by 0.8. Explain his success.

[*Submitted by Alan Wayne*]

(Answers on page 122)

SOLUTIONS

A Cryptarithm

894. [March, 1974] *Proposed by J. A. H. Hunter, Toronto, Canada.*

In this alphametic we naturally have a prime $MATHS$!

$$\begin{array}{r} THIS \\ MAN \\ THIS \\ IS \\ \hline MATHS \end{array}$$

Solution by Martin Moore, Walls, Mississippi.

Solution is:

$$\begin{array}{r} 6749 \\ 132 \\ 6749 \\ 49 \\ \hline 13679 \end{array} \text{— prime.}$$

Also solved by Winifred Asprey, Miguel Bamberger, John A. Beidler, Arthur J. Bradley, Romae J. Cormier, Stephen C. Currier, Jr., Clayton W. Dodge, Thomas E. Elsner, Robert S. Fish, Erica Flapan, Charles D. Friesen, Harry M. Gehman, Norman D. Hardy, Karl Heuer, M. K. King, Vaclav Konecny, Sidney Kravitz, Andrew Langer, Kay P. Litchfield, Edwin McGravy, Joseph Michalowicz, John W. Milsom, Richard O'Beirne, Roland F. Smith, Steven K. Tomlin, R. F. Wardrop, Alan Wayne, Kenneth M. Wilke, and the proposer.

An Integral

Solution by Joseph Silverman, Brown University.

In the following, the information in the boxes on the right indicates the substitution being made.

$$\int \sqrt{\sec^2 x + a}\, dx = \int \sqrt{\frac{u^a + a}{u^a - 1}}\,\frac{1}{u}\, du \qquad \boxed{\begin{array}{l} u = \sec x \\ \dfrac{du}{u\sqrt{u^2 - 1}} = dx \end{array}}$$

$$= \frac{1}{2}\int \sqrt{\frac{z + a}{z - 1}}\,\frac{1}{z}\, dz \qquad \boxed{\begin{array}{l} z = u^2 \\ \tfrac{1}{2}dz = u\,du \end{array}}$$

$$= -(a+1)\int \frac{v^2}{(v^2 + a)(v^2 - 1)}\, dv \qquad \boxed{\begin{array}{l} v^2 = \dfrac{z + a}{z - 1} \\ -2(1+a)\dfrac{v}{(v^2-1)^2}dv = dz \end{array}}$$

$$= \int \left(\frac{1}{2}\,\frac{1}{v + 1} - \frac{1}{2}\,\frac{1}{v - 1} - \frac{a}{v^2 + a}\right) dv$$

$$= \frac{1}{2}\log\left(\frac{v + 1}{v - 1}\right) - \sqrt{a}\tan^{-1}\frac{v}{\sqrt{a}} + C.$$

Substituting back, $\frac{1}{2}\log\left(\dfrac{v + 1}{v - 1}\right) = \log(\sqrt{\sec^2 x + a} + \tan x) - \frac{1}{2}\log(a + 1).$

$$\sqrt{a}\tan^{-1}\frac{v}{\sqrt{a}} = \sqrt{a}\tan^{-1}\frac{\sqrt{\sec^2 x + a}}{\sqrt{a}\tan x} = \sqrt{a}\cos^{-1}\left(\sqrt{\frac{a}{a + 1}}\sin x\right)$$

$$= \sqrt{a}\sin^{-1}\left(\sqrt{\frac{a}{a + 1}}\sin x\right) + \text{constant}.$$

Consolidating constants,

$$\int \sqrt{\sec^2 x + a}\, dx = \log\left(\sqrt{\sec^2 x + a} + \tan x\right) + \sqrt{a}\sin^{-1}\left(\sqrt{\frac{a}{a + 1}}\sin x\right) + C.$$

Also solved by Manghto Ahuja, B. S. Aujla, Miguel Bamberger, V. Chinnaswamy, Santo M. Diano, Alex G. Ferrer, Michael Goldberg, Douglass L. Grant, Bob Hanek, Norman D. Hardy, G. A. Heuer, Vaclav Konecny, T. K. PuttaSwamy, Ellis J. Rich, Fred Safier, E. P. Starke, Edward T. H. Wang, William H. Wertman, Kenneth M. Wilke, Tanakon Ungpiyakul, Qazi Zameeruddin, Aleksandras Zujus, and the proposer.

A Popular Pythagorean Problem

896. [March, 1974] *Proposed by Stephen B. Maurer, Phillips Exeter Academy, New Hampshire.*

A Pythagorean triplet is a triple (a, b, c) of integers such that $a^2 + b^2 = c^2$. Prove that there are infinitely many Pythagorean triplets of the form $(a, a+1, c)$.

I. *Comment by Edward T. H. Wang, Wilfrid Laurier University.*

This problem is certainly not new. It is well known that all Pythagorean triplets (a, b, c) are completely determined by $a = 2uv$, $b = u^2 - v^2$, $c = u^2 + v^2$, where $u > v$. Hence $b = a + 1$ would imply that $u^2 - v^2 - 2uv = 1$. Putting $u = x + y$ and $v = y$ then reduces the last equation to $x^2 - 2y^2 = 1$ which is a special case of the Pellian equation $x^2 - by^2 = 1$ in elementary number theory. Such an equation has infinitely many solutions when b is not a perfect square; this fact, as well as methods for finding all the solutions, have been known since Fermat.

II. *Comment by Michael Goldberg, Washington, D. C.*

Fermat showed that if the integral sides of a right triangle are $(a, a+1, c)$, where $c^2 = 2a^2 + 2a + 1$, then the integral sides of a larger right triangle are $(A, A+1, C)$, where $A = 2c + 3a + 1$ and $C = 3c + 4a + 2$.

III. *Comment by G. E. Bergum, South Dakota State University.*

In a paper, *Pythagorean triangles whose legs differ by one,* Dale C. Koepp of the South Dakota School of Mines and Technology, has shown that the matrix

$$M = \begin{bmatrix} 2 & 1 & 2 \\ 1 & 2 & 2 \\ 2 & 2 & 3 \end{bmatrix}$$

transforms Pythagorean triples into Pythagorean triples while preserving the distance between the legs. Hence $M^n \begin{pmatrix} 3 \\ 4 \\ 5 \end{pmatrix}$ is a Pythagorean triple of the form $(a, a+1, c)$ for $n \geqq 0$.

(*Editor's comment.* Several solvers referenced Sierpinski's *Pythagorean Triangles,* Beiler's *Recreation in the Theory of Numbers* and Olds' *Continued Fractions.*)

Also solved by Winifred Asprey, Gerald E. Bergum, Charles K. Brown, Romae J. Cormier, Stephen C. Currier, Jr., Santo Diano, Clayton W. Dodge, Thomas E. Elsner, Robert S. Fish, Stanley Fox, William F. Fox, Sherman Grable, Bob Hanek, Norman D. Hardy, C. T. Haskell, G. A. Heuer, Harvey J. Hindin, John L. Hunsucker, Kenneth Jackman, Steven Kahan, Vaclav Konecny, Lew Kowarski, Sidney Kravitz, Andrew Langer, Kay P. Litchfield, Graham Lord, John Manning, Gerald R. Martin, James W. McHutchion, Robert A. Meyer, Otto Mond, Danny Myers, Richard C. O'Beirne, F. D. Parker, Bob Prielipp, Lawrence A. Ringenberg, Harry D. Ruderman, Louis Sass, and the proposer.

A Relative Inequality

897. [March, 1974] *Proposed by C. S. Venkataraman, Sree Kerala Varma College, Trichur, South India.*

If $a_1, a_2, a_3, \cdots, a_k$ are the numbers prime to and not greater than n, prove that

$$\sum_{i=1}^{k} \left(\frac{a_i}{n - a_i}\right) \geqq \phi(n).$$

Solution by James W. McHutchion, Columbus, Ohio.

$\phi(n) \leqq n-1$ is a simple inequality easily obtained from the definition of $\phi(n)$. But $(n-1, n) = 1$. Thus

$$\phi(n) \leqq \frac{n-1}{1} = \frac{n-1}{n-(n-1)} \leqq \sum_{i=1}^{k} \frac{a_i}{n - a_i},$$

where $1 \leqq a_i \leqq n$, $(a_i, n) = 1$.

Also solved by Dwight Olson and Kathy Belie, G. E. Bergum, Romae J. Cormier, Santo Diano, Robert F. Fish, Bob Hanek, Vaclav Konecny, Bob Prielipp and N. J. Kuenzi, Henry S. Lieberman, Graham Lord, Arthur Marshall, Robert A. Meyer, Henry J. Ricardo, E. P. Starke, Gillian W. Valk, Edward T. H. Wang, Kenneth M. Wilke, Ira Wolf, and the proposer.

Five Centers in a Triangle

898. [March, 1974] *Proposed by Roger D. H. Jones, University of Georgia.*

There are five points associated with every triangle: the orthocenter, the centroid, the incenter, the circumcenter, and the nine-point center. Prove that if any two of these coincide the triangle is equilateral.

Solution by Graham Lord, Temple University.

The circumcenter, O, the centroid, G, the nine-pont center, N, and the orthocenter, H, colline and are such that $OH = 3OG = 2ON$. Thus if any two of these four points coincide they all do; then the medians will be the altitudes and the triangle will be equilateral. If the incenter, I, is G (resp. H) the angle bisectors are the medians (resp. the altitudes) then the triangle is equilateral. Finally, as $IN = \frac{1}{2}R - r$ and $OI^2 = R^2 - 2Rr$, where R, (r), is the circumradius, (inradius), the incenter coinciding with either N or O would imply $R = 2r$: the triangle is again equilateral.

Also solved by Charles Chouteau, Clayton W. Dodge, Ragnar Dybvik, Alex G. Ferrer, Henry S. Lieberman, Lawrence A. Ringenberg, K. R. S. Sastry, Kenneth M. Wilke, William Wynne Willson, Gregory Wulczyn, and the proposer.

Mean Triangular Twins

899. [March, 1974] *Proposed by Charles W. Trigg, San Diego, California.*

The arithmetic mean of the twin primes 5 and 7 is the triangular number 6. Are there any other twin primes with a triangular mean?

Solution by Daniel Shanks, Bethesda, Maryland.

$$\tfrac{1}{2}(m^2 + m) - 1 = \tfrac{1}{2}(m-1)(m+2)$$

is composite for all $m > 3$. Therefore, (5, 7) is the only prime-pair having a triangular mean.

Generalization. A difference of triangles:

$$\tfrac{1}{2}(n^2 + n) - \tfrac{1}{2}(m^2 + m)$$

always factors:

$$\tfrac{1}{2}(n-m)(n+m+1)$$

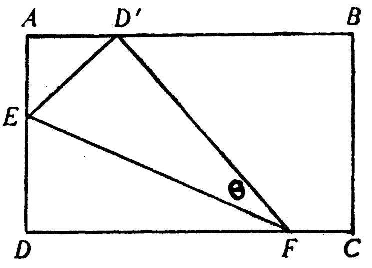
just as a difference of squares does.

Also solved by Leon Palmer and Mangho Ahuja; Joe Albree, Merrill Barnebey, G. E. Bergum, M. K. King and J. R. Boone; Romae J. Cormier, Stephen C. Currier, Jr., Steve Dillard, Clayton W. Dodge, Robert S. Fisk, William F. Fox, Richard A. Gibbs, Marvin Goodman, Susan Greenhaus, Bob Hanek, Steven Hyde, Douglas James, Paul T. Karch and George A. Novacky; Earl E. Keese, Vaclav Konecny, Sidney Kravitz, Bob Prielipp and N. J. Kuenzi; Henry S. Lieberman, Kay P. Litchfield, Graham Lord, James W. McHutchion, Robert A. Meyer, Richard C. O'Beirne, F. D. Parker, C. B. A. Peck, Sidney Penner, Lawrence A. Ringenberg, Gerson B. Robison, M. Rodeen, Louis Sass, K. R. S. Sastry, Joseph Silverman, Paul Smith, Gregg Testini, Steven K. Tomlin, Tanakon Ungpiyakul, Gillian W. Valk, Edward T. H. Wang, Harry L. Whitcomb, Kenneth M. Wilke, Ira Wolf, Sam Zaslavsky, and the proposer.

Not A Centerfold

900. [March, 1974] *Proposed by Murray S. Klamkin, Ford Motor Company, and Seymour Papert, Massachusetts Institute of Technology.*

A long sheet of rectangular paper $ABCD$ is folded such that D falls on AB producing a smooth crease EF with E on AD and F on CD (when unfolded). Determine the minimal area of triangle EFD by elementary methods.



Solution by Michael Goldberg, Washington, D.C.

If $AD = 1$ and K denotes the area of the triangle EFD, then

$$K = (DD')(EF)/4 = (1/\cos\theta)(1/2\cos^2\theta\sin\theta)/4 = \{1/(\sin 2\theta + \tfrac{1}{2}\sin 4\theta)\}/2$$
$$= 1/2M, \text{ where } M = \sin 2\theta + \tfrac{1}{2}\sin 4\theta.$$

Then, $dM/d\theta = 2\cos 2\theta + 2\cos 4\theta = 0$. Hence, $-\cos 4\theta = \cos 2\theta$,

$$\pi - 4\theta = 2\theta,\ \theta = \pi/6 = 30^0,\ K = \{1/(\sqrt{3}/2 + \sqrt{3}/4)\}/2 = 2\sqrt{3}/9 \approx 0.385.$$

The following demonstration can serve as an elementary kinematic solution or verification of the foregoing result. The triangle $D'EF$ attains its extremal area when the line EF intersects its neighboring position at its midpoint G; then the area added by moving the triangle is equal to the area subtracted. As the point D' moves along the straight line AB, the instantaneous center of rotation of the triangle $D'EF$ is on a line through D' perpendicular to AB. Hence, the perpendicular must pass through G. Hence, $AD' = \frac{1}{2}(DF)$, and this occurs only when $\theta = 30°$.

Also solved by Romae J. Cormier, Ralph E. Edwards, Bob Hanek, Donald A. Happel, Robert M. Hashway, Stephen C. Currier, Jr., Jordi Dou, Leonard Goldstone, M. K. King, Vaclav Konecny, Lew Kowarski, Dale C. Mead, Lawrence A. Ringenberg, M. Rodeen, Fred Safier, Edward T. H. Wang, Harry L. Whitcomb, Sam Zaslavsky, and the proposers.

ANSWERS

A614. The given number equals $(10^2 + 1)^4$. Consequently, the sum of the divisors equals $(101^5 - 1)/100$.

A615.
$$1/q = \sum_{n=0}^{\infty} (10^a - q)^n 10^{-(n+1)a}.$$

The pattern reveals itself to some degree in the decimal expansion $1/98 = .01 + .0002 + .000004 + \cdots = .010204081632653061224489795918\cdots$. On the other hand, the expansion 1/3 in the geometric progression bears almost no resemblance to the normal decimal expansion: $1/3 = .1 + .07 + .049 + .0343 + \cdots$.

A616. The volume ratio, $V_t/V_h = 1/2$. The surface ratio, $S_h/S_t = 6/4$. $V_t = S_t r_t/3$ and $V_h = S_h r_h/3$. Therefore the ratio of the radii of the inscribed spheres is $r_t/r_h = (V_t/S_t)/(V_h/S_h) = (V_t/V_h)(S_h/S_t) = 3/4$.

A617. Without loss of generality, $a < b$. There exists a rational number r/s between $a^{1/mn}$ and $b^{1/mn}$. Hence $a < (r/s)^{mn} < b$ or $a < (r^n)^m/(s^m)^n < b$.

A618. More generally, we have $S = \prod_{i=1}^m (1 - x_i)^{n_i} + \prod_{i=1}^m (1 + x_i)^{n_i} \geqq 2$, where $-1 \leqq x_i \leqq 1$, $n_i < 0$ for $i = 1, 2, \cdots, m$. Since $a + b \geqq 2\sqrt{ab}$ for $a, b \geqq 0$, we have $S \geqq 2 \prod_{i=1}^m (1 - x_i^2)^{n_i/2} \geqq 2$ with equality if and only if $x_i = 0$.

A619. $a + bc = (a + b)(a + c)$ if and only if either $a + b + c = 1$ or $a = 0$

(Quickies on page 116–117)

THE CARUS MATHEMATICAL MONOGRAPHS

The Monographs are a series of expository books intended to make topics in pure and applied mathematics accessible to teachers and students of mathematics and also to non-specialists and scientific workers in other fields.

These numbers are currently available:

1. *Calculus of Variations,* by G. A. Bliss.
2. *Analytic Functions of a Complex Variable,* by D. R. Curtiss.
3. *Mathematical Statistics,* by H. L. Rietz.
4. *Projective Geometry,* by J. W. Young.
6. *Fourier Series and Orthogonal Polynomials,* by Dunham Jackson.
7. *Vectors and Matrices,* by C. C. MacDuffee.
8. *Rings and Ideals,* by N. H. McCoy.
9. *The Theory of Algebraic Numbers,* by Harry Pollard.
10. *The Arithmetic Theory of Quadratic Forms,* by B. W. Jones.
11. *Irrational Numbers,* by Ivan Niven.
12. *Statistical Independence in Probability, Analysis and Number Theory,* by Mark Kac.
13. *A Primer of Real Functions* (Second edition), by Ralph P. Boas, Jr.
14. *Combinatorial Mathematics,* by H. J. Ryser.
15. *Noncommutative Rings,* by I. N. Herstein.
16. *Dedekind Sums,* by Hans Rademacher and Emil Grosswald.
17. *The Schwarz Function and its Applications,* by Philip J. Davis.

One copy of each Carus Monograph may be purchased by individual members of the Association for $5.00 each; additional copies and copies for nonmembers are priced at $10.00 each.

Orders with remittance should be sent to:

MATHEMATICAL ASSOCIATION OF AMERICA
1225 Connecticut Avenue, N.W.
Washington, D.C. 20036

MATHEMATICS MAGAZINE, Vol. 48, No. 2, MARCH–APRIL 1975